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กึ่งกรุปไตรภาคที่บรรจุ CT-ไอดีล

On Ternary Semigroups Containing CT-Ideals

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บทคัดย่อ

ในบทความวิจัยนี้ เราเสนอแนวคิดของ CT-ไอดีลบนกึ่งกรุปไตรภาค เราศึกษาคุณสมบัติที่น่าสนใจของ CT-ไอดีล และศึกษาความสัมพันธ์ระหว่างไอดีลสองด้านแท้และ CT-ไอดีล สุดท้ายนี้เราแสดงว่า ถ้าไอดีลสองด้าน A เป็นกึ่งกรุปย่อยไตรภาคปรกติ ของกึ่งกรุปไตรภาคปรกติ S แล้ว ทุกๆ CT-ไอดีลของไอดีลสองด้าน A เป็น CT-ไอดีลของกึ่งกรุปไตรภาคปรกติ S

คำสำคัญ : กึ่งกรุปไตรภาค ; กึ่งกรุปไตรภาคปรกติ ; ไอดีลสองด้าน ; CT-ไอดีล

Abstract

In this article, we give a definition of the CT-ideal in a ternary semigroup. We study some interesting properties of the CT-ideal and the relationship between the proper two-sided ideal and the CT-ideal. Finally, we show that if a two-sided ideal A is a regular ternary subsemigroup of a regular ternary semigroup S, then every CT-ideal of A is also a CT-ideal of S.

Keywords : ternary semigroup ; regular ternary semigroup ; two-sided ideal ; CT-ideal

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Introduction

The notion of a ternary semigroup generalized the notion of a ternary group and was introduced by Lehmer (1932) as follows: a *ternary semigroup* is a non-empty set S together with a ternary operation $[\ ]: S\times S\times S\to S,$ written as $(x_1,x_2,x_3)\mapsto [x_1x_2x_3],$ satisfying the following identity

$$[[x_1x_2x_3]x_4x_5]=[x_1[x_2x_3x_4]x_5]=[x_1x_2[x_3x_4x_5]] \text{ for all } x_1,x_2,x_3,x_4,x_5\in S.$$

Throughout this paper $S=(S,[\ ])$ denotes a ternary semigroup with respect to the ternary operation $[\ ]$ unless otherwise stated. For non-empty subsets A,B and C of a ternary semigroup $S,$ we denote

$$[ABC]:=\{[abc]\mid a\in A, b\in B, c\in C\}.$$

If $A=\{a\},$ we write $[\{a\}BC]$ as $[aBC].$ Any other cases can be defined analogously. A non-empty subset A of a ternary semigroup S is called a *ternary subsemigroup* (Dixit & Dewan, 1995) of S if $[AAA]\subseteq A.$

Example 1. (Dixit & Dewan, 1995) Let $S=\{-i,0,i\}.$ Then $(S,[\ ])$ is a ternary semigroup under the multiplication over complex numbers while $(S,\cdot)$ is not a semigroup under complex numbers multiplication.

The notion of ideal play very important role to study the algebraic structures. Sioson (1965) studied ideal theory in ternary semigroups as follows: a non-empty subset I of a ternary semigroup S is called a *left (right) ideal* of S if $[SSI]\subseteq I$ $([ISS]\subseteq I).$ If I is a left and a right ideal of $S,$ then I is called a *two-sided ideal* of $S.$ A two-sided ideal I of a ternary semigroup S is said to be *proper* if $I\neq S.$ A proper two-sided ideal I of a ternary semigroup S is said to be *maximal* if for any two-sided ideal A of S such that $I\subseteq A\subseteq S,$ then $I=A$ or $A=S.$ Note that the union of two two-sided ideals of S is a two-sided ideal of $S,$ and the intersection of two two-sided ideals of $S,$ if it is non-empty, is a two-sided ideal of $S.$ For any element a of a ternary semigroup $S,$ the intersection of all two-sided ideals of S containing a of S is called the *principal two-sided ideal* of S generated by a and denoted by $I(a).$ Then the principal two-sided ideal generated by a is given by Sioson (1965), and it is of the form $I(a)=\{a\}\cup[SSa]\cup[aSS]\cup[SSaSS].$ Moreover, Sioson (1965) gave the notion of regular ternary semigroup as follows: a ternary semigroup S is said to be *regular* if for each $a\in S,$ there exist elements x,y in S such that $[axaya]=a.$

Fabrici (1984) introduced the concept of the covered ideal of a semigroup, which is a proper ideal I of a semigroup S satisfying $I\subseteq S(S\setminus I)S,$ and obtained some properties in terms of maximal ideals. In 2016, Changphas and Summaprab (2016) discussed the structure of ordered semigroups containing covered ideals. Later, Omidi and Davvaz (2017) discussed the notion of a covered Γ-hyperideal and characterized the properties of covered Γ-hyperideal in ordered Γ-semihypergroups. Recently, the notion of covered hyperideals was introduced and discussed in ordered semihypergroups by Ali and Khan (2023). In this article, we introduce the concept of covered two-sided ideals (shortly: CT -ideals) of ternary semigroups and study the structure of ternary semigroups containing CT -ideal.

Methods

In this research, we shall use the methods of proof in mathematics and the concept of covered two sided-ideal of a ternary semigroup. In this section, we give the definition and examples of covered two-sided ideals of a ternary semigroup.

Definition 1. A proper two-sided ideal I of a ternary semigroup S is called a *covered two-sided ideal* (shortly: CT *-ideal*) of S if $I \subseteq [SS(S \setminus I)] \cup [(S \setminus I)SS] \cup [SS(S \setminus I)SS]$.

Example 2. Let $S = \{a, b, c, d, e\}$. Define the ternary operation $[\ ]$ on S by, for all $x, y, z \in S$, $[xyz] = x * (y * z)$ where $*$ is the binary operation on S defined by:

$*$	a	b	c	d	e
a	a	b	a	d	a
b	a	b	a	d	a
c	a	b	a	d	a
d	a	b	a	d	a
e	a	b	a	d	a

Then $(S, [\ ])$ is a ternary semigroup (Kar *et al.,* 2020). We have $\{a, b, d\}$, $\{a, b, c, d\}$, $\{a, b, d, e\}$ and S are two-sided ideals of S, and we have $\{a, b, d\}$ is a CT -ideal of S. But $\{a, b, c, d\}$ and $\{a, b, d, e\}$ are not CT -ideals of S.

Example 3. Let $S = \{0, a, b, c, 1\}$. Define the ternary operation $[\ ]$ on S by, for all $x, y, z \in S$, $[xyz] = x * (y * z)$ where $*$ is the binary operation on S defined by:

$*$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	a	a
b	0	0	b	b	b
c	0	0	b	c	c
1	0	a	b	c	1

Then $(S, [\ ])$ is a ternary semigroup (Bashir & Shabir, 2009). We have $\{0\}$, $\{0, a\}$, $\{0, b\}$, $\{0, a, b\}$, $\{0, a, b, c\}$ and S are two-sided ideals of S, and $\{0\}$, $\{0, a\}$, $\{0, b\}$, $\{0, a, b\}$ and $\{0, a, b, c\}$ are CT -ideals of S. Note that the union and the intersection of two CT -ideals of a ternary semigroup S is a CT -ideal. Then, we will prove in the following section.

Results

We begin this section with the following theorem characterizes when a proper two-sided ideal of a ternary semigroup is not a CT -ideal.

Theorem 1. Let A and B be any two different proper two-sided ideals of a ternary semigroup S such that $A \cup B = S.$ Then none of them is CT -ideal of $S.$

Proof. Let A and B be any two different proper two-sided ideals of S such that $A \cup B = S.$ Since $A \cup B = S,$ we have $S \setminus A \subseteq B$ and $S \setminus B \subseteq A.$ If A is a CT -ideal of $S,$ it follows that

$$A \subseteq [SS(S \setminus A)] \cup [(S \setminus A)SS] \cup [SS(S \setminus A)SS] \subseteq [SSB] \cup [BSS] \cup [[SSB]SS] \subseteq B \cup B \cup [BSS] \subseteq B.$$

Since $A \cup B = S$ and $A \subseteq B,$ we obtain that $S = B.$ This is a contradiction. Thus, A is not a CT -ideal of $S.$ Similarly, we can show that B is not a CT -ideal of $S.$

Corollary 2. If a ternary semigroup S contains two or more maximal two-sided ideals, then none of them is a CT -ideal of $S.$

Proof. Suppose that A and B are two different maximal two-sided ideals of $S.$ We claim that $A \cup B$ is a two-sided ideal of $S.$ Since A and B are two-sided ideals of $S,$ it follows that

$$[SS(A \cup B)] \subseteq [SSA] \cup [SSB] \subseteq A \cup B \text{ and } [(A \cup B)SS] \subseteq [ASS] \cup [BSS] \subseteq A \cup B.$$

Hence, $A \cup B$ is a two-sided ideal of S and $A \subset A \cup B.$ Since A is a maximal two-sided ideal of $S,$ we obtain that $A \cup B = S.$ Thus, by Theorem 1, neither A nor B is a CT -ideal of $S.$

Theorem 3. Let A and B be two any CT -ideals of a ternary semigroup S such that $A \cup B \neq S.$ Then $A \cup B$ is a CT -ideal of $S.$

Proof. From Corollary 2, we know that $A \cup B$ is a two-sided ideal of $S.$ Since $A \cup B \neq S,$ then $A \cup B$ is a proper two-sided ideal of $S.$ We will show that $A \cup B$ is a CT -ideal of $S.$ By assumption, we have

$$A \subseteq [SS(S \setminus A)] \cup [(S \setminus A)SS] \cup [SS(S \setminus A)SS] \text{ and } B \subseteq [SS(S \setminus B)] \cup [(S \setminus B)SS] \cup [SS(S \setminus B)SS].$$

Let $a \in A \cup B.$ If $a \in A,$ we have $a \in A \subseteq [SS(S \setminus A)] \cup [(S \setminus A)SS] \cup [SS(S \setminus A)SS],$ and so $a \in [SSx_1] \cup [x_2SS] \cup [SSx_3SS]$ for some $x_1, x_2, x_3 \in S \setminus A.$ Now, we consider eight following cases:

Case 1: $x_1, x_2, x_3 \in S \setminus (A \cup B).$ We obtain that

$$a \in [SSx_1] \cup [x_2SS] \cup [SSx_3SS] \subseteq [SS(S \setminus (A \cup B))] \cup [(S \setminus (A \cup B))SS] \cup [SS(S \setminus (A \cup B))SS].$$

Case 2: $x_1, x_2, x_3 \in (S \setminus A) \cap B.$ Then $x_1, x_2, x_3 \in B.$ We have

$$x_1, x_2, x_3 \in B \subseteq [SS(S \setminus B)] \cup [(S \setminus B)SS] \cup [SS(S \setminus B)SS].$$

Then there exist $y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9 \in S \setminus B$ such that $x_1 \in [SSy_1] \cup [y_2SS] \cup [SSy_3SS]$, $x_2 \in [SSy_4] \cup [y_5SS] \cup [SSy_6SS]$ and $x_3 \in [SSy_7] \cup [y_8SS] \cup [SSy_9SS]$. If $y_1, y_2, y_3 \in A$, we obtain that $x_1 \in [SSy_1] \cup [y_2SS] \cup [SSy_3SS] \subseteq [SSA] \cup [ASS] \cup [SSASS] \subseteq A \cup A \cup [ASS] \subseteq A$. Thus, $x_1 \in A$, which is a contradiction as $x_1 \in S \setminus A$. Hence, $y_1, y_2, y_3 \in S \setminus A$. Similarly, we obtain that $y_4, y_5, y_6, y_7, y_8, y_9 \in S \setminus A$. Thus, $y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9 \in (S \setminus A) \cap (S \setminus B) = S \setminus (A \cup B)$. We consider

$$\begin{aligned}
a &\in [SSx_1] \cup [x_2SS] \cup [SSx_3SS] \\
&\subseteq [SS[[SSy_1] \cup [y_2SS] \cup [SSy_3SS]]] \cup [[[SSy_4] \cup [y_5SS] \cup [SSy_6SS]]SS] \cup [SS[[SSy_7] \cup [y_8SS] \cup [SSy_9SS]]SS] \\
&= [[SSS]Sy_1] \cup [SSy_2SS] \cup [[SSS]Sy_3SS] \cup [SSy_4SS] \cup [y_5S[SSS]] \cup [SSy_6S[SSS]] \\
&\quad \cup [[SSS]Sy_7SS] \cup [SSy_8S[SSS] \cup [[SSS]Sy_9S[SSS]] \\
&\subseteq [SSy_1] \cup [SSy_2SS] \cup [SSy_3SS] \cup [SSy_4SS] \cup [y_5SS] \cup [SSy_6SS] \cup [SSy_7SS] \cup [SSy_8SS] \cup [SSy_9SS] \\
&\subseteq [SS(S \setminus (A \cup B))] \cup [SS(S \setminus (A \cup B))SS] \cup [SS(S \setminus (A \cup B))SS] \cup [SS(S \setminus (A \cup B))SS] \cup [(S \setminus (A \cup B))SS] \\
&\quad \cup [SS(S \setminus (A \cup B))SS] \cup [SS(S \setminus (A \cup B))SS] \cup [SS(S \setminus (A \cup B))SS] \cup [SS(S \setminus (A \cup B))SS] \\
&= [SS(S \setminus (A \cup B))] \cup [(S \setminus (A \cup B))SS] \cup [SS(S \setminus (A \cup B))SS].
\end{aligned}$$

Case 3: $x_1, x_2 \in S \setminus (A \cup B)$ and $x_3 \in (S \setminus A) \cap B$. Since $x_3 \in B$, we have that $z_1, z_2, z_3 \in S \setminus B$ such that $x_3 \in [SSz_1] \cup [z_2SS] \cup [SSz_3SS]$. If $z_1, z_2, z_3 \in A$, then

$$x_3 \in [SSz_1] \cup [z_2SS] \cup [SSz_3SS] \subseteq [SSA] \cup [ASS] \cup [SSASS] \subseteq A \cup A \cup [ASS] \subseteq A.$$

Thus, $x_3 \in A$, which is a contradiction as $x_3 \in S \setminus A$. Hence, $z_1, z_2, z_3 \in S \setminus A$. So, we obtain that $z_1, z_2, z_3 \in (S \setminus A) \cap (S \setminus B) = S \setminus (A \cup B)$. Thus,

$$\begin{aligned}
a &\in [SSx_1] \cup [x_2SS] \cup [SSx_3SS] \\
&\subseteq [SS(S \setminus (A \cup B))] \cup [(S \setminus (A \cup B))SS] \cup [SS[[SSz_1] \cup [z_2SS] \cup [SSz_3SS]]SS] \\
&= [SS(S \setminus (A \cup B))] \cup [(S \setminus (A \cup B))SS] \cup [[SSS]Sz_1SS] \cup [SSz_2S[SSS]] \cup [[SSS]Sz_3S[SSS]] \\
&\subseteq [SS(S \setminus (A \cup B))] \cup [(S \setminus (A \cup B))SS] \cup [SSz_1SS] \cup [SSz_2SS] \cup [SSz_3SS] \\
&\subseteq [SS(S \setminus (A \cup B))] \cup [(S \setminus (A \cup B))SS] \cup [SS(S \setminus (A \cup B))SS] \cup [SS(S \setminus (A \cup B))SS] \cup [SS(S \setminus (A \cup B))SS] \\
&= [SS(S \setminus (A \cup B))] \cup [(S \setminus (A \cup B))SS] \cup [SS(S \setminus (A \cup B))SS].
\end{aligned}$$

Case 4: $x_1, x_3 \in S \setminus (A \cup B)$ and $x_2 \in (S \setminus A) \cap B$. Then this similar to case 3.

Case 5: $x_2, x_3 \in S \setminus (A \cup B)$ and $x_1 \in (S \setminus A) \cap B$. Then this similar to case 3.

Case 6: $x_1 \in S \setminus (A \cup B)$ and $x_2, x_3 \in (S \setminus A) \cap B$. Then this similar to case 3.

Case 7: $x_2 \in S \setminus (A \cup B)$ and $x_1, x_3 \in (S \setminus A) \cap B$. Then this similar to case 3.

Case 8: $x_3 \in S \setminus (A \cup B)$ and $x_1, x_2 \in (S \setminus A) \cap B$. Then this similar to case 3.

In all these cases, we obtain that $a \in [SS(S \setminus (A \cup B))] \cup [(S \setminus (A \cup B))SS] \cup [SS(S \setminus (A \cup B))SS]$, and so $A \subseteq [SS(S \setminus (A \cup B))] \cup [(S \setminus (A \cup B))SS] \cup [SS(S \setminus (A \cup B))SS]$. Similarly, we can show that if $a \in B$, then $B \subseteq [SS(S \setminus (A \cup B))] \cup [(S \setminus (A \cup B))SS] \cup [SS(S \setminus (A \cup B))SS]$. Hence,

$$A \cup B \subseteq [SS(S \setminus (A \cup B))] \cup [(S \setminus (A \cup B))SS] \cup [SS(S \setminus (A \cup B))SS].$$

This shows that $A \cup B$ is a CT -ideal of S.

Theorem 4. Let A be a two-sided ideal of a ternary semigroup S and B be a CT -ideal of S. Then $A \cap B$ is a CT -ideal of S.

Proof. Let $x, y \in A$ and $z \in B$. To show that $A \cap B$ is a CT -ideal of S. Since A and B are two-sided ideals of S, we have $[xyz] \in [AAB] \subseteq [SSB] \subseteq B$ and $[xyz] \in [AAB] \subseteq [ASS] \subseteq A$. Thus, $[xyz] \in A \cap B$, and so $\varnothing \neq A \cap B \subseteq S$. Next, we consider $[SS(A \cap B)] \subseteq [SSA] \subseteq A$ and $[SS(A \cap B)] \subseteq [SSB] \subseteq B$. Similarly, $[(A \cap B)SS] \subseteq [ASS] \subseteq A$ and $[(A \cap B)SS] \subseteq [BSS] \subseteq B$. Thus, $[SS(A \cap B)] \subseteq A \cap B$ and $[(A \cap B)SS] \subseteq A \cap B$ Hence, $A \cap B$ is a two sided-ideal of S. Since B is a CT -ideal of S, then $A \cap B$ is a proper two sided-ideal of S and so

$$\begin{aligned} A \cap B \subseteq B &\subseteq [SS(S \setminus B)] \cup [(S \setminus B)SS] \cup [SS(S \setminus B)SS] \\ &\subseteq [SS(S \setminus (A \cap B))] \cup [(S \setminus (A \cap B))SS] \cup [SS(S \setminus (A \cap B))SS]. \end{aligned}$$

Hence, $A \cap B$ is a CT -ideal of S.

The next corollary is an immediate consequence of Theorem 4 since every CT-ideal is a two sided ideal.

Corollary 5. If A and B are CT -ideals of a ternary semigroup S, then $A \cap B$ is a CT -ideal of S.

Theorem 6. If A is a two-sided ideal of a ternary semigroup S such that $A \subset [SSx] \cup [xSS] \cup [SSxSS]$ and $A \neq [SSx] \cup [xSS] \cup [SSxSS]$ for some $x \in S$, then A is a CT -ideal of S.

Proof. Assume that A is a two-sided ideal of a ternary semigroup S such that $A \subset [SSx] \cup [xSS] \cup [SSxSS]$ and $A \neq [SSx] \cup [xSS] \cup [SSxSS]$ for some $x \in S$. If $x \in A$, then

$$A \subset [SSx] \cup [xSS] \cup [SSxSS] \subseteq [SSA] \cup [ASS] \cup [[SSA]SS] \subseteq A \cup A \cup [ASS] \subseteq A.$$

Thus, $A = [SSx] \cup [xSS] \cup [SSxSS]$, which is a contradiction. Hence, $x \in S \setminus A$, and so

$$A \subset [SSx] \cup [xSS] \cup [SSxSS] \subseteq [SS(S \setminus A)] \cup [(S \setminus A)SS] \cup [SS(S \setminus A)SS].$$

Therefore, A is a CT -ideal of S.

Corollary 7. A ternary semigroup S in which x does not belongs to $[SSx] \cup [xSS] \cup [SSxSS]$ contains a CT -ideal.

Proof. Let $A = [SSx] \cup [xSS] \cup [SSxSS]$. We claim that A is a two-sided ideal of S. Consider

$$[SSA] = [SS[[SSx] \cup [xSS] \cup [SSxSS]]]$$

$$= [[SSS]Sx] \cup [SSxSS] \cup [[SSS]SxSS]$$
$$\subseteq [SSx] \cup [SSxSS] \cup [SSxSS] \subseteq A$$

and
$$[ASS] = [[[SSx] \cup [xSS] \cup [SSxSS]]SS]$$
$$= [SSxSS] \cup [xS[SSS]] \cup [SSxS[SSS]]$$
$$\subseteq [SSxSS] \cup [xSS] \cup [SSxSS] \subseteq A.$$

Then A is a two-sided ideal of S. If $x \notin A$, we obtain that

$$A = [SSx] \cup [xSS] \cup [SSxSS] \subseteq [SS(S \setminus A)] \cup [(S \setminus A)SS] \cup [SS(S \setminus A)SS].$$

Thus, A is a CT -ideal of S.

Theorem 8. Let S be a ternary semigroup. If S contains proper two-sided ideals such that there is no any two proper two-sided ideals in which their intersection is empty, then S contains at least one CT -ideal.

Proof. Assume that S contains proper two-sided ideals such that there is no any two proper two-sided ideals in which their intersection is empty. Then S contains a proper two-sided ideal A. We claim that $B = [SS(S \setminus A)] \cup [(S \setminus A)SS] \cup [SS(S \setminus A)SS]$ is a two-sided ideal of S. Consider

$$[SSB] = [SS[[SS(S \setminus A)] \cup [(S \setminus A)SS] \cup [SS(S \setminus A)SS]]$$
$$= [[SSS]S(S \setminus A)] \cup [SS(S \setminus A)SS] \cup [[SSS]S(S \setminus A)SS]$$
$$\subseteq [SS(S \setminus A)] \cup [SS(S \setminus A)SS] \cup [SS(S \setminus A)SS] \subseteq B$$

and
$$[BSS] = [[[SS(S \setminus A)] \cup [(S \setminus A)SS] \cup [SS(S \setminus A)SS]]SS]$$
$$= [SS(S \setminus A)SS] \cup [(S \setminus A)S[SSS]] \cup [SS(S \setminus A)S[SSS]]$$
$$\subseteq [SS(S \setminus A)SS] \cup [(S \setminus A)SS] \cup [SS(S \setminus A)SS] \subseteq B.$$

Then B is a two-sided ideal of S. By assumption, $A \cap B \neq \varnothing$. Thus, $A \cap B$ is a proper two-sided ideal of S. Since $A \cap B \subseteq A$, it follows that $S \setminus A \subseteq S \setminus (A \cap B)$, and so

$$A \cap B \subseteq B = [SS(S \setminus A)] \cup [(S \setminus A)SS] \cup [SS(S \setminus A)SS]$$
$$\subseteq [SS(S \setminus (A \cap B))] \cup [(S \setminus (A \cap B))SS] \cup [SS(S \setminus (A \cap B))SS].$$

This implies $A \cap B$ is a CT -ideal of S.

Let S be a ternary semigroup. A proper two-sided ideal I of S is called the *greatest two-sided ideal* of S if it contains every proper two-sided ideal of S. If a ternary semigroup S contains the greatest two-sided ideal, we denote it by I^*. The following theorem characterizes the relationship between the proper two-sided ideal and the CT -ideal.

Theorem 9. Let S be a ternary semigroup which satisfies just one of the following conditions:

(1) S contains the greatest two-sided ideal I^* which is a CT -ideal of S.

(2) S is a regular ternary semigroup and for any proper two-sided ideal B and for every principal two-sided ideal $I(a) \subseteq B,$ there exists $b \in S \setminus B$ such that $I(a) \subseteq I(b).$

Then every proper two-sided ideal of S is a CT -ideal of $S.$

Proof. Let A be a proper two-sided ideal of $S.$ First, suppose that the condition (1) holds. Then $A \subseteq I^*$ and $S \setminus I^* \subseteq S \setminus A.$ Since I^* is a CT -ideal of $S,$ it follows that

$$A \subseteq I^* \subseteq [SS(S \setminus I^*)] \cup [(S \setminus I^*)SS] \cup [SS(S \setminus I^*)SS] \subseteq [SS(S \setminus A)] \cup [(S \setminus A)SS] \cup [SS(S \setminus A)SS].$$

Hence, A is a CT -ideal of $S.$

Secondary, suppose that the condition (2) holds. Clearly, $[SSS] \subseteq S.$ Let $x \in S.$ Since S is a regular, then there exists $y, z \in S$ such that

$$x = [xyxzx] = [xyxz[xyxzx]] = [[xyx][zxy][xzx]] \in [[SSS][SSS][SSS]] \subseteq [SSS].$$

Thus, $S \subseteq [SSS]$ and so $S = [SSS].$ Next, let B be a proper two-sided ideal of S and $a \in B.$ Thus, $I(a) \subseteq B.$ Then there exists $b \in S \setminus B$ such that $I(a) \subseteq I(b) \subseteq S.$ Since $S = [SSS],$ we obtain that $S = [SSS] = [SS[SSS]] = [SSSSS].$ Thus, $b = [x_1x_2x_3x_4x_5]$ for some $x_1, x_2, x_3, x_4, x_5 \in S.$ If $x_3 \in B,$ it follows that

$$b = [x_1x_2x_3x_4x_5] \in [SSBSS] = [[SSB]SS] \subseteq [BSS] \subseteq B.$$

This contradicts to $b \in S \setminus B.$ Hence, $x_3 \in S \setminus B,$ and so

$$b = [x_1x_2x_3x_4x_5] \in [SS(S \setminus B)SS] \subseteq [SS(S \setminus B)] \cup [(S \setminus B)SS] \cup [SS(S \setminus B)SS].$$

It follows that $I(b) \subseteq [SS(S \setminus B)] \cup [(S \setminus B)SS] \cup [SS(S \setminus B)SS].$ Thus,

$$a \in I(a) \subseteq I(b) \subseteq [SS(S \setminus B)] \cup [(S \setminus B)SS] \cup [SS(S \setminus B)SS].$$

Hence, $B \subseteq [SS(S \setminus B)] \cup [(S \setminus B)SS] \cup [SS(S \setminus B)SS].$ Therefore, B is a CT -ideal of $S.$

The following example illustrates Theorem 9.

Example 4. Let $S = \{0,1,2,3,4,5\}.$ Define the ternary operation [ ] on S by, for all $x, y, z \in S,$ $[xyz] = (x * y) * z$ where $*$ is the binary operation on S defined by:

$*$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	1	1	1
2	0	1	2	3	1	1
3	0	1	1	1	2	3
4	0	1	4	5	1	1
5	0	1	1	1	4	5

Then $(S,[\ ])$ is a ternary semigroup (Shabir & Bano, 2008). One can check that S is a regular ternary semigroup. We have the sets $\{0\}$ and $\{0,1\}$ are proper two-sided ideals of S. One can also check that S contains the greatest two-sided ideal $I^* = \{0,1\}$ and I^* is a CT-ideal of S. Thus, the proper two-sided ideals $\{0\}$ and $\{0,1\}$ are CT-ideals of S. Moreover, consider the proper two-sided ideal $B = \{0,1\}$ of S. We have $I(0) = \{0\} \subseteq B$ and $I(1) = \{0,1\} \subseteq B$, and there exists $2 \in S \setminus B$ such that $I(0), I(1) \subseteq I(2) = S$. Thus, B is a CT-ideal of S.

We now present the main result of this paper.

Theorem 10. Let A be a two-sided ideal of a regular ternary semigroup S. If A is a regular ternary subsemigroup of S, then any CT-ideal B of A is also a CT-ideal of S.

Proof. Assume that A is a regular ternary subsemigroup of S. Suppose that B is a CT-ideal of A. We claim that B is a two-sided ideal of S. Let $a \in B \subseteq A$ and $x, y \in S$. Then $[axy] \in [BSS] \subseteq [ASS] \subseteq A$ and $[xya] \in [SSB] \subseteq [SSA] \subseteq A$. We set $a_1 = [axy] \in A$ and $a_2 = [xya] \in A$. Since A is a regular ternary subsemigroup of S, then there exist $b_1, b_2, b_3, b_4 \in A$ such that $a_1 = [a_1 b_1 a_1 b_2 a_1]$ and $a_2 = [a_2 b_3 a_2 b_4 a_2]$. Thus,

$$\begin{aligned} a_1 = [a_1 b_1 a_1 b_2 a_1] &= [[axy] b_1 [axy] b_2 [axy]] \\ &\in [[BSS]A[BSS]A[BSS]] \\ &\subseteq [[B[SSA]AS[SSS]] \\ &\subseteq [[B[SSA][ASS]] \\ &\subseteq [BAA] \\ &\subseteq B \end{aligned}$$

and

$$\begin{aligned} a_2 = [a_2 b_3 a_2 b_4 a_2] &= [[xya] b_3 [xya] b_4 [xya]] \\ &\in [[SSB]A[SSB]A[SSB]] \\ &\subseteq [[SSS]S[SSA][ASS]B] \\ &\subseteq [[SSA][ASS]B] \\ &\subseteq [AAB] \\ &\subseteq B. \end{aligned}$$

Hence, $a_1, a_2 \in B$. This shows that B is a two-sided ideal of S. Since B is a CT-ideal of A, then $B \subseteq [AA(A \setminus B)] \cup [(A \setminus B)AA] \cup [AA(A \setminus B)AA]$. Since $B \subset A \subseteq S$, $\varnothing \neq A \setminus B \subseteq S \setminus B$. It follows that

$$B \subseteq [AA(A \setminus B)] \cup [(A \setminus B)AA] \cup [AA(A \setminus B)AA] \subseteq [SS(S \setminus B)] \cup [(S \setminus B)SS] \cup [SS(S \setminus B)SS].$$

Therefore, B is a CT-ideal of S.

Example 5. Let $S=\{a,b,c,d,e\}$. Define the ternary operation $[\;]$ on S by, for all $x,y,z\in S$, $[xyz]=x*(y*z)$ where $*$ is the binary operation on S defined by:

$*$	a	b	c	d	e
a	a	a	c	d	a
b	a	b	c	d	a
c	a	a	c	d	a
d	a	a	c	d	a
e	a	a	c	d	e

Then $(S,[\;])$ is a ternary semigroup (Kar *et al.*, 2020). One can check that S is regular. Consider the subset $A=\{a,c,d,e\}$ of S. One can also check that A is a two-sided ideal and A is a regular ternary subsemigroup of S. Moreover, we have $B=\{a,c,d\}$ is a CT-ideal of A. Thus, by Theorem 10, B is also a CT-ideal of S.

Discussion

In this research, we present the results for CT-ideal of a ternary semigroup. The results of this research are to extend the results obtained by Fabrici (1984) (semigroups) to ternary semigroups. The notion of CT-ideal of a ternary semigroup is a proper two- sided ideal I of a ternary semigroup S such that $I\subseteq[SS(S\setminus I)]\cup[(S\setminus I)SS]\cup[SS(S\setminus I)SS]$, which is a generalization of the notion of covered ideal of a semigroup ($I\subseteq S(S\setminus I)S$). It is observed that the notion of CT-ideal of a ternary semigroup is more interesting than the notion of covered ideal of a semigroup. In our future study, we will extend the notion of CT-ideals to the structure of ordered ternary semigroups.

Conclusions

Analogous to the theory of semigroups, the algebraic structure of ternary semigroups and its related properties are studied by many mathematicians in various topics. Sioson (1965) studied an ideal theory in ternary semigroups and defined regular ternary semigroups. Thongkam and Changphas (2015) introduced and studied the concept of two-sided bases of ternary semigroups. In this research, we study CT-ideals of ternary semigroups and investigate some properties of the CT-ideal and the relationship between the proper two-sided ideal and the CT-ideal. Finally, we prove in Theorem 10 that every CT-ideal of a two-sided ideal A such that A is a regular ternary subsemigroup of a regular ternary semigroup S is also a CT-ideal of a regular ternary semigroup S.

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