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สมการไดโอแฟนไทน์ $p^{2x}+q^{y}=z^{4}$ และ $p^{2x}-q^{y}=z^{4}$ เมื่อ p และ q เป็นจำนวนเฉพาะ

On the Diophantine Equations $p^{2x}+q^{y}=z^{4}$ and $p^{2x}-q^{y}=z^{4}$ where p and q are Primes

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บทคัดย่อ

ในงานวิจัยนี้ได้ศึกษาสมการไดโอแฟนไทน์ $p^{2x}+q^{y}=z^{4}$ และ $p^{2x}-q^{y}=z^{4}$ เมื่อ p และ q เป็นจำนวนเฉพาะ พบว่า สมการไดโอแฟนไทน์ $p^{2x}+q^{y}=z^{4}$ มีผลเฉลยทั้งหมดที่เป็นจำนวนเต็มที่ไม่เป็นลบคือ $(p,q,x,y,z)\in\{(3,7,1,1,2)\}\cup\left\{\left(p,2,1,\log_2(p+1)+2,\sqrt{p+2}\right)\middle|\ \log_2(p+1),\sqrt{p+2}\in\mathbb{Z}\right\}\cup\{(2,17,3,1,3)\}$ และสมการไดโอแฟนไทน์ $p^{2x}-q^{y}=z^{4}$ มีผลเฉลยทั้งหมดที่เป็นจำนวนเต็มที่ไม่เป็นลบคือ $(p,q,x,y,z)\in\left\{\left(p,q,1,\log_q(2p-1),\sqrt{p-1}\right)\middle|\ \log_q(2p-1),\sqrt{p-1}\in\mathbb{Z}\right\}\cup$ $\left\{\left(p,2,1,\log_2(p-1)+2,\sqrt{p-2}\right)\middle|\ \log_2(p-1),\sqrt{p-2}\in\mathbb{Z}\right\}\cup\{(p,q,0,0,0)\}\cup\left\{(p,p,u,2u,0)\middle|u\in\mathbb{Z}^{+}\right\}$

คำสำคัญ : สมการไดโอแฟนไทน์ ; ข้อคาดการณ์ของกาตาลัน

Abstract

In this paper, we study Diophantine equations $p^{2x}+q^{y}=z^{4}$ and $p^{2x}-q^{y}=z^{4}$, where p and q are primes. We found that all non-negative integer solutions of the Diophantine equation $p^{2x}+q^{y}=z^{4}$ are of the following $(p,q,x,y,z)\in\{(3,7,1,1,2)\}\cup\left\{\left(p,2,1,\log_2(p+1)+2,\sqrt{p+2}\right)\middle|\ \log_2(p+1),\sqrt{p+2}\in\mathbb{Z}\right\}$ $\cup\{(2,17,3,1,3)\}$ and all non-negative integer solutions of the Diophantine equation $p^{2x}-q^{y}=z^{4}$ are of the following $(p,q,x,y,z)\in\left\{\left(p,q,1,\log_q(2p-1),\sqrt{p-1}\right)\middle|\ \log_q(2p-1),\sqrt{p-1}\in\mathbb{Z}\right\}\cup$ $\left\{\left(p,2,1,\log_2(p-1)+2,\sqrt{p-2}\right)\middle|\ \log_2(p-1),\sqrt{p-2}\in\mathbb{Z}\right\}\cup\{(p,q,0,0,0)\}\cup\left\{(p,p,u,2u,0)\middle| u\in\mathbb{Z}^{+}\right\}$.

Keywords : Diophantine equation ; Catalan's Conjecture

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Introduction

A Diophantine equation is an equation in which only integer solution is allowed. Many research studies about Diophantine equations are ancient. However, no general methods for finding a solution of a given equation exist. The well-known Diophantine equation of the form $a^x + b^y = z^2$ has been studied by many researchers. For example, Chotchaisthit (2012) found all non-negative integer solutions of the Diophantine equation $4^x + p^y = z^2$. Later, Chotchaisthit (2013a) showed that $(3,0,3)$ is the only non-negative integer solution of the Diophantine equation $2^x + 11^y = z^2$, $(p,x,y,z) = (7,0,1,3)$ and $(p,x,y,z) = (3,2,2,5)$ are the only two solutions of the Diophantine equation $p^x + (p+1)^y = z^2$, where x, y and z are non-negative integers and p is a Mersenne prime (Chotchaisthit, 2013b). Many related research was solved as seen in literature (Suvarnamani et al., 2011; Singha, 2021; Sroysang, 2012; Bacani & Rabago, 2015; Burshtein, 2018; Dokchan & Pakapongpun, 2021; Mina & Bacani, 2021)

In 2017, Burshtein (2017) showed for all primes $p \geq 2$ and $y = 1$, that the Diophantine equation $p^x + q^y = z^4$ has infinitely many positive integer solutions (x,q,z).

Later, Burshtein (2021) proved that the Diophantine equations $p^4 \pm q^y = z^4$ have no solution, when p, q are distinct primes, and y, z are positive integers.

Inspired by the works mentioned earlier (Burshtein, 2017, 2019, 2020, 2021; Mina & Bacani, 2019), we will find all non-negative integer solutions (x,y,z) of the Diophantine equations $p^{2x} \pm q^y = z^4$, where p and q are primes.

Methods

In this section, we give some helpful Theorems for this study.

Theorem 1 (Catalan's conjecture) (Mihailescu, 2004) $(3,2,2,3)$ is the only solution (a,b,x,y) to the Diophantine equation $a^x - b^y = 1$, where a, b, x and y are integers with $\min\{a,b,x,y\} > 1$.

Theorem 2 (Burshtein, 2021) Let y and z be positive integers. For all three possibilities

(a) $p = 2$ and q an odd prime,

(b) p an odd prime and $q = 2$,

(c) p, q distinct odd primes,

the equation $p^4 + q^y = z^4$ has no solution.

Theorem 3 (Burshtein, 2021) Let y and z be positive integers. For all three possibilities

(a) $p = 2$ and q an odd prime,

(b) p an odd prime and $q=2$,

(c) p,q distinct odd primes,

the equation $p^4 - q^y = z^4$ has no solution.

Inspired by Burshtein's results, we are interested in finding all non-negative integer solutions of the Diophantine equations $p^{2x} \pm q^y = z^4$.

Results

In this section, we find all non-negative integer solutions (x,y,z) of the Diophantine equations $p^{2x} \pm q^y = z^4$, where p and q are prime numbers.

First, we consider the Diophantine equation $p^{2x} + q^y = z^4$, where p and q are prime numbers.

Theorem 4. For any prime numbers p and q, let

$A = \{(3,7,1,1,2)\}$,

$B = \{(p,2,1,\log_2(p+1)+2,\sqrt{p+2}) \mid \log_2(p+1), \sqrt{p+2} \in \mathbb{Z}\}$,

$C = \{(2,17,3,1,3)\}$.

Then $(p,q,x,y,z) \in A \cup B \cup C$ are all non-negative integer solutions of the Diophantine equation $p^{2x} + q^y = z^4$.

Proof. Let x, y and z be non-negative integers such that (p,q,x,y,z) is a solution of the Diophantine equation $p^{2x} + q^y = z^4$. Since $z^4 - p^{2x} = q^y$, we have $(z^2 - p^x)(z^2 + p^x) = q^y$. Then there exists a non-negative integer u such that $z^2 - p^x = q^u$ and $z^2 + p^x = q^{y-u}$. Thus $2p^x = q^u(q^{y-2u} - 1)$, which implies $y - 2u \ge 1$. Moreover, it is clear that $\gcd(q^u, q^{y-2u} - 1) = 1$.

Consider the following cases:

Case 1: $u = 0$. We have $z^2 - p^x = 1$. If $x = 0$, then $z^2 = 2$, which is a contradiction. If $x = 1$, then $z^2 - p = 1$ and $2p + 1 = q^y$, so $p = z^2 - 1$. Thus $z = 2, p = 3, q = 7$ and $y = 1$. Thus $(p,q,x,y,z) \in A$.

If $x > 1$, it is easy to check that $z > 1$. By Catalan's conjecture, we get $z = 3, p = 2$ and $x = 3$. Since $z^2 + p^x = q^y$, it implies that $q = 17$ and $y = 1$. Then $(p,q,x,y,z) \in C$.

Case 2: $u > 0$.

Case 2.1: $p = 2$. We have $2^{x+1} = q^u(q^{y-2u} - 1)$. Since $\gcd(q^u, q^{y-2u} - 1) = 1$ and $u > 0$, we get $q^u = 2^{x+1}$. It follows that $q = 2$ and $u = x + 1$. So $z^2 = 2^x + 2^{x+1} = 3 \cdot 2^x$ which contradicts the fact that z is an integer.

Case 2.2 : $p \ne 2$. Thus $\gcd(2, p^x) = 1$. We consider the following 2 cases.

Case 2.2.1: $2 = q^{y-2u} - 1$ and $p^x = q^u$. If $p \neq q$, then $x = u = 0$, so $z^2 = 1 + 1 = 2$, which is a contradiction. If $p = q \neq 2$, then $x = u$, so $z^2 = 2q^u$, which is impossible.

Case 2.2.2: $2 = q^u$ and $p^x = q^{y-2u} - 1$. We get $q = 2$ and $u = 1$, so $p^x = 2^{y-2} - 1$.

If $x = 0$, then $2^{y-2} = 2$ which implies that $y = 3$. Hence $z^2 = 3$, which is a contradiction.

If $x = 1$, then $2^{y-2} = p + 1$, so $y = \log_2(p+1) + 2$. We also get that $z^2 = p + 2$, i.e.; $z = \sqrt{p+2}$. Then $(p,q,x,y,z) \in B$.

In case $x > 1$, we have $y - 2 > 1$ and $2^{y-2} - p^x = 1$, so it is a contradiction to Catalan's conjecture. This finishes the proof.

Example 1. Let $p = 7$ and $q = 2$. We have $\log_2(p+1) + 2 = 5$ and $\sqrt{p+2} = 3$. By Theorem 4, the Diophantine equation $7^{2x} + 2^y = z^4$ has only one non-negative integer solution $(x,y,z) = (1,5,3)$.

Next, we consider the Diophantine equation $p^{2x} - q^y = z^4$, where p and q are prime numbers.

Theorem 5. For any prime numbers p and q, let

$$A = \left\{ \left(p,q,1,\log_q(2p-1),\sqrt{p-1}\right) \mid \log_q(2p-1),\sqrt{p-1} \in \mathbb{Z} \right\},$$

$$B = \left\{ \left(p,2,1,\log_2(p-1)+2,\sqrt{p-2}\right) \mid \log_2(p-1),\sqrt{p-2} \in \mathbb{Z} \right\},$$

$$C = \left\{ (p,q,0,0,0) \right\},$$

$$D = \left\{ (p,p,u,2u,0) \mid u \in \mathbb{Z}^+ \right\}.$$

Then $(p,q,x,y,z) \in A \cup B \cup C \cup D$ are all non-negative integer solutions of the Diophantine equation $p^{2x} - q^y = z^4$.

Proof. Let x, y and z be non-negative integers such that (p,q,x,y,z) is a solution of the Diophantine equation $p^{2x} - q^y = z^4$. Thus $\left(p^x - z^2\right)\left(p^x + z^2\right) = q^y$. Then there exists a non-negative integer u such that $p^x - z^2 = q^u$ and $p^x + z^2 = q^{y-u}$, so $y \geq 2u$ and $2p^x = q^u\left(q^{y-2u} + 1\right)$.

Consider the following cases:

Case 1: $u = 0$. We have $z^2 = p^x - 1$.

If $z = 0$, then $x = 0$ and $y = 0$, so $(p,q,x,y,z) \in C$.

Let $z \geq 1$. We get $x \geq 1$. If $x = 1$, then $z^2 = p - 1$, so $z = \sqrt{p-1}$.

Since $q^y = p^x + z^2 = 2p - 1$, we have $y = \log_q(2p-1)$. Hence $(p,q,x,y,z) \in A$.

In case of $x > 1$, we get $z > 1$ and $p^x - z^2 = 1$, which contradicts Catalan's conjecture.

Case 2: $u>0$. If $y-2u=0$, then $y=2u$, so $2p^x=2q^u$. Thus $(p,q,x,y,z)\in D$.

Let $y-2u\geq 1$. We have $\gcd\left(q^u,q^{y-2u}+1\right)=1$. Obviously, $p\neq 2$. Since $2p^x=q^u\left(q^{y-2u}+1\right)$, the only possible case is $q^u=2$ and $p^x=q^{y-2u}+1$. Thus $q=2$ and $u=1$. It follows that $p^x=2^{y-2}+1$. We get $x\geq 1$.

If $x=1$, then $p=2^{y-2}+1$, so $y=\log_2(p-1)+2$ and $z=\sqrt{p-2}$. Then $(p,q,x,y,z)\in B$.

Let $x>1$. It follows that $y-2>1$. By Catalan's conjecture and the fact that $p^x-2^{y-2}=1$, we get $p=3, x=2$ and $y-2=3$. Hence $z^2=7$, which is a contradiction. This completes the proof.

Example 2. Let $p=3$ and $q=2$. We have $\log_2(p-1)+2=3$ and $\sqrt{p-2}=1$. By Theorem 5, the Diophantine equation $3^{2x}-2^y=z^4$ has only two non-negative integer solutions $(x,y,z)=(1,3,1)$ and $(x,y,z)=(0,0,0)$.

Example 3. Let $p=2$ and $q=3$. We have $\log_q(2p-1)=1$ and $\sqrt{p-1}=1$. By Theorem 5, the Diophantine equation $2^{2x}-3^y=z^4$ has only two non-negative integer solutions $(x,y,z)=(1,1,1)$ and $(x,y,z)=(0,0,0)$.

Discussion

In this paper, we obtain all non-negative integer solutions (x,y,z) of the Diophantine equations $p^{2x}+q^y=z^4$ and $p^{2x}-q^y=z^4$, where p and q are prime numbers. A possible generalization of our results is to find all integral solutions (x,y,z) of the Diophantine equations $p^x\pm q^y=z^4$, where p and q are prime numbers.

Conclusions

In this paper, we get all non-negative integer solutions (x,y,z) of the Diophantine equations $p^{2x}+q^y=z^4$ and $p^{2x}-q^y=z^4$, where p and q are prime numbers.

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