ผลเฉลยวางนัยทั่วไปของสมการโคชี-ออยเลอร์อันดับ 2 และอันดับ 3 โดยใช้การแปลงเอลซากิ

The Generalized Solutions of Second and Third-Order Cauchy-Euler Equations by Using the Elzaki Transforms

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บทคัดย่อ

บทความนี้มีวัตถุประสงค์เพื่อศึกษาผลเฉลยวางนัยทั่วไปของสมการโคชี-ออยเลอร์ที่อยู่ในรูปแบบของ $t^2 y''(t) + aty'(t) + by(t) = 0$ และ $t^3 y'''(t) + at^2 y''(t) + bty'(t) + cy(t) = 0$ เมื่อ a, b และ c เป็นจำนวนเต็ม และ $t \in \mathbb{R}$ โดยใช้เทคนิคการแปลงเอลซากิ ผลเฉลยที่ได้จะอยู่ในปริภูมิดิสทริบิวชัน ชนิดของผลเฉลยอยู่ในรูปแบบผลเฉลยดิสทริบิวชัน $y(t) = \delta^{(k)}(t)$ และผลเฉลยแบบอ่อน $y(t) = \dfrac{t^k H(t)}{k!}$ โดยชนิดของผลเฉลยขึ้นอยู่กับเงื่อนไขค่าของ a, b และ c

คำสำคัญ : สมการโคชี-ออยเลอร์ ; ฟังก์ชันไดแรคเดลต้า ; การแปลงเอลซากิ ; ผลเฉลยวางนัยทั่วไป

Abstract

This paper aims to study the generalized solutions of Cauchy-Euler equations of the form $t^2 y''(t) + aty'(t) + by(t) = 0,$ and $t^3 y'''(t) + at^2 y''(t) + bty'(t) + cy(t) = 0,$ where $a, b,$ and c are integers and $t \in R,$ using Elzaki transform technique. The solutions are in the space of distributions. Types of solutions are in the form of a distributional solution $y(t) = \delta^{(k)}(t)$ and a weak solution $y(t) = \dfrac{t^k H(t)}{k!}$ which depends on the values of $a, b,$ and $c.$

Keywords : Cauchy-Euler equation ; Dirac delta function ; Elzaki transform ; The generalized solutions

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Introduction

The differential operator L is defined by

$$\begin{aligned} Lt &= \left[a_n(x)\frac{d^n}{dx^n} + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \ldots + a_1(x)\frac{d}{dx} + a_0(x) \right] t \\ &= \sum_{m=0}^{n} a_m(x)\frac{d^m t}{dx^m}, \end{aligned} \tag{1}$$

where the coefficients $a_n(x)$ are infinitely differentiable functions.

The solution differential equations are in form

$$Lt = \sum_{m=0}^{n} a_m(x)\frac{d^m t}{dx^m} = \tau , \tag{2}$$

where τ is an arbitrary known distribution. A distribution t is a solution of (2) if for every test function ϕ , then

$$\langle Lt , \phi \rangle = \langle \tau , \phi \rangle. \tag{3}$$

It is well known that the fundamental solution is the solution for $\tau = \delta(x)$. In searching for a solution t to the differential equation (2), the following can be considered

1. The solution t is a sufficiently smooth function, so that the operation in (2) can be performed in the classical sense, and the resulting equation is an identity. Then t is the classical solution.

2. The solution t is not sufficiently smooth, so that the operation in (2) cannot be performed, but it is satisfied (3) as a distribution. It is then a weak solution.

3. The solution t is a singular distribution and satisfied (3). It is then a distributional solution.

All these solutions are called generalized solutions, see (Kanwal, 2004).

The integral transform method is widely used to solve several differential equations with initial or boundary conditions, see (Elzaki & Elzaki , 2011). To solve differential equations, the integral transform is extensively used, and thus there are several works on the theory and application of integral transforms, such as the Laplace, Fourier, Mellin, and Hankel, to name but a few. The Laplace transform method plays a worthy role in solving linear ordinary differential equations and corresponding initial value problems, having the idea of replacing operations of calculus on functions with operations of algebra on transforms, see (Elzaki & Elzak , 2011; Ghil & Kim, 2015). Elzaki's transform is applied for initial value problems with variable coefficients and for solving integral equations of convolution type. This transform is a special case of the Laplace transform, see (Alderremy, 2018).

In 2011, T. M. Elzaki and S. M. Elzaki discussed some relationships between the Laplace transform and the Elzaki transform. They obtained solutions of first and second-order ordinary differential equations using both transforms and showed that the Elzaki transform is closely connected with the Laplace transform (Elzaki & Elzaki, , 2011).

In 2013, H. Kim studied the solution of a differential equation with variable coefficients of the second kind by using the Elzaki Transforms (Kim, 2013).

In 2015, B. Ghil and H. Kim studied solutions of

$$t^2 y''(t) + at y'(t) + by(t) = 0 \tag{4}$$

where $a, b \in Z$, and $t \in \mathbb{R}$, by using the Laplace transform technique (Ghil & Kim, 2015).

In 2016, H. Kim checked the method to find the basis of the Euler-Cauchy equation by transforming. The most common form is the second order of the form (4) (Kim, 2016).

In 1999, Kananthai studied the distribution solutions of

$$t^3 y'''(t) + t^2 y''(t) + ty'(t) + my(t) = 0, \tag{5}$$

where $m \in Z$, and $t \in \mathbb{R}$. He found that solutions of (5), which are either the distributional solutions or the classical solutions, depend on the values of m (Kananthai, 1999).

In 2019, S. Jhanthanam *et al.* studied the generalized solutions of

$$t^3 y'''(t) + at^2 y''(t) + bt y'(t) + cy(t) = 0, \tag{6}$$

where $a, b, c \in \mathbb{Z}$, and $t \in \mathbb{R}$, by using the Laplace transform technique. They obtained the solutions of (6), which are either the distributional solutions or the classical solution, depending on the values of a, b and c (Jhanthanam *et al.*, 2019).

In 2018, H. Kim et al. studied the distribution solutions of

$$at^3 y'''(t) + bt^2 y''(t) + cty'(t) + dy(t) = 0, \tag{7}$$

where a, b, c are real constants with $a \neq 0$ and $t \in \mathbb{R}$. They obtained solutions of (7), which are either the distributional solutions or the classical solutions (Kim *et al*., 2018).

In this paper, we study the generalized solutions of Cauchy-Euler equations by using the Elzaki transforms, for second and third-order. Elzaki transform is one such technique to solve differential equations with initial conditions and has been effectively used in solving the linear differential equation. The goal of this work is to determine the solutions to such equations in the space of distributions.

Preliminaries

Before proceeding to our main results, the following definitions, Theorems and concepts are required.

Definition 1 A distribution T is a continuous linear functional on the space $\mathscr{D}$ of a real -valued functions with infinitely and bounded support. The space of all such distribution is denoted by $\mathscr{D}'$.

For every $T \in \mathscr{D}'$ and $\phi \in \mathscr{D}$, the value that T acts on ϕ is denoted by $\langle T, \phi \rangle$. Now ϕ called a test function in $\mathscr{D}$.

Definition 2 (The Differentiation of Distribution). The k -order derivative of a distribution T, is denote by

$$\left\langle T^{(k)}, \phi \right\rangle = (-1)^k \left\langle T, \phi^{(k)} \right\rangle \text{ for all } \phi \in \mathscr{D}.$$

Example 1 Let $\delta(t)$ be the Dirac delta function, then

$$\left\langle \delta^{(n)}, \phi \right\rangle = (-1)^n \left\langle \delta, \phi^{(n)} \right\rangle = (-1)^n \phi^{(n)}(0).$$

Definition 3 (The Multiplication of a Distribution by an Infinitely Differentiable Function). Let $\alpha(t)$ be an infinitely differentiable function. Then, the product of $\alpha(t)$ with any distribution T in $\mathscr{D}'$ is defined by $\langle \alpha T, \phi \rangle = \langle T, \alpha\phi \rangle$ for all $\phi \in \mathscr{D}$.

Definition 4 Let c be a real number, $A \in \mathbb{R}$ and $f(t)$ be a locally integrable function which satisfies the following condition:

(i) $f(t) = 0$ all $t < A$;

(ii) There exists c such that $e^{-t/c} f(t)$ is absolutely integrable over $\mathbb{R}$.

The Elzaki transform of $f(t)$ is defined by

$$E\left[f(t)\right] = T(u) = u\int_{A}^{\infty} e^{-t/u} f(t)\,dt, \tag{8}$$

where $u \in (k_1, k_2)$ and $k_1, k_2 > 0$.

It is known that if $f(t)$ is continuous, then $T(u)$ is an analytic function on the half-plane $\mathrm{R}(u) > \sigma_a$, where σ_a is an abscissa of absolute convergence for $E\left[f(t)\right]$.

Lemma 1 (Elzaki & Elzaki , 2012), Let $T(u)$ be the Elzaki transform of $f(t)$ such that

(i) $uT\left(\frac{1}{u}\right)$ is a meromorphic function, with singularities having $R(u) < c,$ and

(ii) there exists a circular region with radius R and positive constants M and K with

$$\left|uT\left(\frac{1}{u}\right)\right| < MR^{-K}.$$

Then the function $f(t)$ defined by

$$f(t) = E^{-1}\left[T(u)\right] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{ut}\,uT\left(\frac{1}{u}\right)dt = \sum residues \quad of \left[e^{ut}\,uT\left(\frac{1}{u}\right)\right] \tag{9}$$

Definition 5 Let c be a real number and $f(t)$ be a distribution satisfying the following properties:

(i) $f(t)$ is a right-sided distribution, that is $f(t) \in \mathscr{D}'_{\mathrm{R}}$.

(ii) There exists c such that $e^{-t/c} f(t)$ is a tempered distribution.

The Elzaki transform $f(t)$ satisfying (ii) is defined by

$$T(u) = E\left[f(t)\right] = \left\langle e^{-t/c} f(t), Y(t)ue^{-\left(\frac{1}{u}-\frac{1}{c}\right)t} \right\rangle, \tag{10}$$

where $Y(t)$ is an infinitely differentiable function with support bounded on the left, which equals to 1 over a neighborhood of the support of $f(t)$.

For $R(u) < c,$ the function $Y(t)ue^{-\left(\frac{1}{u}-\frac{1}{c}\right)t}$ is a testing function in the space of tempered distributions S and $ue^{-t/u} f(t)$ is in the space S'. Equation (10) can be reduced to

$$T(u) = E[f(t)] = \left\langle f(t), ue^{-t/u} \right\rangle. \tag{11}$$

Lemma 2 (Kanwal, 2004), Let $\varphi(t)$ be an infinitely differentiable function. Then

$$\varphi(t)\delta^{(m)}(t) = (-1)^m \varphi^{(m)}(0)\delta(t) + (-1)^{m-1} m\,\varphi^{(m-1)}(0)\delta'(t) + (-1)^{m-2}\frac{m(m-1)}{2!}\varphi^{(m-2)}(0)\delta''(t) + \ldots + \varphi(0)\delta^{(m)}(t). \tag{12}$$

A useful formula that follows from (12), for any monomial $\varphi(t) = t^n$, is

$$t^n\delta^{(m)}(t) = \begin{cases} 0, & m < n; \\ (-1)^n \dfrac{m!}{(m-n)!}\delta^{(m-n)}(t), & m \ge n. \end{cases} \tag{13}$$

Theorem 1 (Elzaki *et al*., 2012), Let $T(u)$ be the Elzaki transform of $f(t)$, $E[f(t)] = T(u)$. Then

$$E\left[f^{(n)}(t)\right] = \frac{T(u)}{u^n} - \sum_{k=0}^{n-1} u^{2-n+k} f^{k}(0),$$

for $n \in \mathbb{N}$.

Theorem 2 We can naturally obtain the following results from the definition and simple calculations.

(i) $E[tf(t)] = u^2 \dfrac{d}{du}T(u) - uT(u),$

(ii) $E[f'(t)] = \dfrac{T(u)}{u} - uf(0),$

(iii) $E[tf'(t)] = u^2 \dfrac{d}{du}\left(\dfrac{T(u)}{u} - uf(0)\right) - u\left(\dfrac{T(u)}{u} - uf(0)\right),$

(iv) $E\left[t^2 f(t)\right] = u^4 \dfrac{d^2}{du^2}T(u),$

(v) $E\left[t^2 f''(t)\right] = u^4 \dfrac{d^2}{du^2}\left(\dfrac{T(u)}{u^2} - f(0) - uf'(0)\right),$

(vi) $E\left[t^3 f(t)\right] = u^6 \dfrac{d^3}{du^3}T(u) + 3u^5 \dfrac{d^2}{du^2}T(u),$

(vii) $E\left[t^3 f'''(t)\right] = u^6 \dfrac{d^3}{du^3}\left(\dfrac{T(u)}{u^3} - \dfrac{f(0)}{u} - f'(0) - uf''(0)\right)$

$$+3u^5 \frac{d^2}{du^2}\left(\frac{T(u)}{u^3}-\frac{f(0)}{u}-f'(0)-uf''(0)\right).$$

Proof. For the proof of (i) and (iii), see (Elzaki & Elzaki, 2011), (ii), see (Elzaki, 2011), (v), see (Devi *et al*., 2017), (iv) and (vi), see (Abbasbandy & Eslaminasab, 2015) for more details.

Now we show proof for (vii). From (vi) and Theorem 1, we have

$$\begin{aligned} E\left[t^3 f'''(t)\right] &= u^6 \frac{d^3}{du^3} E\left[f'''(t)\right] + 3u^5 \frac{d^2}{du^2} E\left[f'''(t)\right] \\ &= u^6 \frac{d^3}{du^3}\left(\frac{T(u)}{u^3}-\frac{f(0)}{u}-f'(0)-uf''(0)\right) \\ &\quad + 3u^5 \frac{d^2}{du^2}\left(\frac{T(u)}{u^3}-\frac{f(0)}{u}-f'(0)-uf''(0)\right). \end{aligned}$$ □

Theorem 3 If $f(t)$ is a piecewise continuous function for $t \geq 0$ and has exponential order at infinity with $\left|f(t)\right| \leq Me^{kt}$ where M is constant, $k > 0$, then for any real number $a \geq 0$,

$$E\left[f(t-a)H(t-a)\right] = e^{-a/u}T(u), \tag{14}$$

where $H(t-a)$ is the Heaviside function and $H(0) = 1$.

Proof. From (8), we have

$$\begin{aligned} e^{-a/u}T(u) &= e^{-a/u}u\int_0^\infty e^{-s/u} f(s)ds \\ &= u\int_0^\infty e^{-(s+a)/u} f(s)ds. \end{aligned}$$

Let $t = s + a$, we get

$$\begin{aligned} e^{-a/u}T(u) &= u\int_a^\infty e^{-t/u} f(t-a)dt \\ &= u\int_0^\infty e^{-t/u} f(t-a)H(t-a)dt \\ &= E\left[f(t-a)H(t-a)\right]. \end{aligned}$$ □

By the similar way, we have $E\left[H(t-a)\right] = u^2 e^{-a/u}$, see (Devi *et al.*, 2017).

Theorem 4 Let $H(t)$ be the Heaviside function and $H(0) = 1$, then

$$E\left[\frac{t^k H(t)}{k!}\right] = u^{k+2}, \tag{15}$$

where $k \in \mathbb{N} \cup \{0\}$.

Proof. By equation (11), the desired result was derived.

$$\begin{aligned} E\left[\frac{t^k H(t)}{k!}\right] &= \left\langle \frac{t^k H(t)}{k!}, ue^{-t/u} \right\rangle \\ &= \frac{u}{k!}\int_0^\infty t^k e^{-t/u} dt \\ &= u^{k+2}. \end{aligned}$$

Theorem 5 Let $\delta(t)$ be the Dirac delta function, then

$$E\left[\delta^{(k)}(t)\right] = u^{1-k}, \tag{16}$$

where $k \in \mathbb{N} \cup \{0\}$.

Proof. By (11) and Example 1, it is easy to see that

$$\begin{aligned} E\left[\delta^{(k)}(t)\right] &= \left\langle \delta^{(k)}(t), ue^{-t/u} \right\rangle \\ &= (-1)^k \left\langle \delta(t), u(e^{-t/u})^{(k)} \right\rangle \\ &= u^{1-k}. \end{aligned}$$

□

Methods

By taking the Elzaki transform to $t^2 y''(t) + aty'(t) + by(t) = 0$ on both sides. Let the solution be in the form $T(u) = u^r$, where r is real constant. Case (i) let $r = 1-k$ where $k \in \mathbb{N} \cup \{0\}$, for conditions $(k+2)(k+1) = (k+1)a - b$, then $T(u) = u^{1-k}$. Taking the inverse Elzaki transform to $T(u)$, then distributional solutions are in the form $y(t) = \delta^{(k)}(t)$. Case (ii) let $r = k+2$ where $k \in \mathbb{N} \cup \{0\}$, for conditions $k - k^2 = ka + b$, then $T(u) = u^{k+2}$. Taking the inverse Elzaki transform to $T(u)$, then weak solutions are in the form $y(t) = \dfrac{t^k H(t)}{k!}$.

For third-order Cauchy-Euler equation of the form $t^3 y'''(t) + at^2 y''(t) + bty'(t) + cy(t) = 0$, the solutions can be examined in a similar way as for second-order Cauchy-Euler equation.

Results

This section will state our main results and give their proof.

Theorem 6 Consider the second-order Cauchy-Euler equation of the form

$$t^2 y''(t) + a t y'(t) + b y(t) = 0, \tag{17}$$

where a, b are integers and $t \in \mathbb{R}$. Then the solution to (17) is as follows:

(i) if
$$(k+2)(k+1) = (k+1)a - b, \tag{18}$$

then the solutions of (17) are distributional solutions in the form $\delta^{(k)}(t)$ for $k \in \mathbb{N} \cup \{0\}$;

(ii) if
$$k - k^2 = ka + b, \tag{19}$$

then the solutions of (17) are weak solutions in the form $\dfrac{t^k H(t)}{k!}$ for $k \in \mathbb{N} \cup \{0\}$.

Proof. By taking the Elzaki transform to (17) on both sides and using Theorem 2 (iii) and (v), we have

$$E\left[t^2 y''(t)\right] + aE\left[ty'(t)\right] + bE\left[y(t)\right] = 0$$

$$u^4 \frac{d^2}{du^2}\left(\frac{T(u)}{u^2} - y(0) - uy'(0)\right) + a\left[u^2 \frac{d}{du}\left(\frac{T(u)}{u} - uy(0)\right) - u\left(\frac{T(u)}{u} - uy(0)\right)\right] + bT(u) = 0$$

$$u^2 T''(u) + (a-4)uT'(u) + (b - 2a + 6)T(u) = 0. \tag{20}$$

Let the solution of (20) be in the form $T(u) = u^r$ where r is a real constant.

Substituting $T(u)$, $T'(u)$ and $T''(u)$ in to (20) yields

$$r(r-1)u^r + (a-4)ru^r + (b-2a+6)u^r = 0$$
$$r^2 + (a-5)r + b - 2a + 6 = 0 . \qquad (21)$$

Now, we consider case (i), $r = 1-k$ where $k \in \mathbb{N} \cup \{0\}$. Then (21) becomes

$$(1-k)^2 + (a-5)(1-k) + b - 2a + 6 = 0$$
$$(k+1)(k+2) = (k+1)a - b. \qquad (22)$$

Thus the solution of (20) is $T(u) = u^{1-k}$, for $k \in \mathbb{N} \cup \{0\}$. By taking the inverse Elzaki transform to $T(u)$ and by Theorem 5, we obtain the solution of (17) are distributional solutions in the form

$$y(t) = \delta^{(k)}(t) \qquad (23)$$

for $k \in \mathbb{N} \cup \{0\}$.

Finally, we consider case (ii), $r = k+2$ where $k \in \mathbb{N} \cup \{0\}$. Then (21) becomes

$$(k+2)^2 + (a-5)(k+2) + b - 2a + 6 = 0$$
$$k - k^2 = ka + b. \qquad (24)$$

Thus the solution of (20) is $T(u) = u^{k+2}$, for $k \in \mathbb{N} \cup \{0\}$. By taking the inverse Elzaki transform to $T(u)$ and by Theorem 4, we obtain the solution of (17) are weak solutions in the form

$$y(t) = \frac{t^k H(t)}{k!} \qquad (25)$$

for $k \in \mathbb{N} \cup \{0\}$. □

Example 2 From Theorem 6, if $a = 3$ and $b = 1$, then (17) becomes

$$t^2 y''(t) + 3ty'(t) + y(t) = 0. \qquad (26)$$

It follows form of (18), then the solution of (26) is

$$y(t) = \delta(t). \tag{27}$$

Example 3 From Theorem 6, if $a = 21$ and $b = 100$, then (17) becomes

$$t^2 y''(t) + 21ty'(t) + 100y(t) = 0. \tag{28}$$

It follows form of (18), then the solution of (28) is

$$y(t) = \delta^{(9)}(t). \tag{29}$$

Example 4 From Theorem 6, if $a = -1$ and $b = 1$, then (17) becomes

$$t^2 y''(t) - ty'(t) + y(t) = 0. \tag{30}$$

It follows form of (19), then the solution of (30) is

$$y(t) = tH(t). \tag{31}$$

Example 5 From Theorem 6, if $a = -3$ and $b = 3$, then (17) becomes

$$t^2 y''(t) - 3ty'(t) + 3y(t) = 0, \tag{32}$$

It follows form of (19), then the solution of (32) is

$$y(t) = \frac{t^3 H(t)}{3!}. \tag{33}$$

Theorem 7 Consider the third order Cauchy-Euler equation of the form

$$t^3 y'''(t) + at^2 y''(t) + bty'(t) + cy(t) = 0, \tag{34}$$

where a, b, c are integer and $t \in \mathbb{R}$. Then the solution to (34) is a follows:

(i) if $$(k+3)(k+2)(k+1) = (k+2)(k+1)a - (k+1)b + c, \tag{35}$$

then the solutions of (34) are distributional solutions in the form $\delta^{(k)}(t)$ for $k \in \mathbb{N} \cup \{0\}$;

(ii) if $$k\,(k-2)(k-1) = -k(k-1)a - kb - c, \tag{36}$$

then the solutions of (34) are weak solutions in the form $\dfrac{t^k H(t)}{k!}$ for $k \in \mathbb{N} \cup \{0\}$.

Proof. By taking the Elzaki transform to (34) on both sides and using Theorem 2 (iii), (v) and (vii), we have

$$E\left[t^3 y'''(t)\right] + aE\left[t^2 y''(t)\right] + bE\left[ty'(t)\right] + cE\left[y(t)\right] = 0$$

$$u^6 \frac{d^3}{du^3}\left(\frac{T(u)}{u^3} - \frac{y(0)}{u} - y'(0) - uy''(0)\right) + 3u^5 \frac{d^2}{du^2}\left(\frac{T(u)}{u^3} - \frac{y(0)}{u} - y'(0) - uy''(0)\right)$$

$$+ au^4 \frac{d^2}{du^2}\left(\frac{T(u)}{u^2} - y(0) - uy'(0)\right) + b\left[u^2 \frac{d}{du}\left(\frac{T(u)}{u} - uy(0)\right) - u\left(\frac{T(u)}{u} - uy(0)\right)\right] + cT(u) = 0$$

$$u^3 T'''(u) + (a-6)u^2 T''(u) + (18 - 4a + b)uT'(u) + (6a - 2b + c - 24)T(u) = 0. \tag{37}$$

Let solution of (37) be in the form $T(u) = u^r$, where r is real constant.

Substituting $T(u)$, $T'(u)$, $T''(u)$ and $T'''(u)$ in to (36) leads to

$$r(r-1)(r-2)u^r + (a-6)r(r-1)u^r + (18 - 4a + b)ru^r + (6a - 2b + c - 24)u^r = 0$$

$$r^3 + (a-9)r^2 + (26 - 5a + b)r + 6a - 2b + c - 24 = 0 \tag{38}$$

Now, we consider case (i), $r = 1 - k$ where $k \in \mathbb{N} \cup \{0\}$. Then (38) becomes

$$(1-k)^3+(a-9)(1-k)^2+(26-5a+b)(1-k)+6a-2b+c-24=0$$
$$(k+3)(k+2)(k+1)=(k+2)(k+1)a-(k+1)b+c. \quad (39)$$

Thus the solution of (37) is $T(u)=u^{1-k}$, for $k\in\mathbb{N}\cup\{0\}$. By taking the inverse Elzaki transform to $T(u)$ and by Theorem 5, we obtain the solution of (34) are distributional solutions in the form

$$y(t)=\delta^{(k)}(t) \quad (40)$$

for $k\in\mathbb{N}\cup\{0\}$.

Finally, we consider case (ii), $r=k+2$ where $k\in\mathbb{N}\cup\{0\}$. Then (38) becomes

$$(k+2)^3+(a-9)(k+2)^2+(26-5a+b)(k+2)+6a-2b+c-24=0$$
$$k\ (k-2)(k-1)=-k(k-1)a-kb-c. \quad (41)$$

Thus the solution of (37) is $T(u)=u^{k+2}$, for $k\in N\cup\{0\}$. By taking the inverse Elzaki transform to $T(u)$ and by Theorem 4, we obtain the solution of (34) are weak solutions in the form

$$y(t)=\frac{t^kH(t)}{k!} \quad (42)$$

for $k\in\mathbb{N}\cup\{0\}$. □

Example 6 From Theorem 7, if $a=9, b=19$ and $c=8$, then (34) becomes

$$t^3y'''(t)+9t^2y''(t)+19ty'(t)+8y(t)=0. \quad (43)$$

It follows form of (35), such that the solution of (43) is

$$y(t)=\delta'(t). \quad (44)$$

The solution is similar as example 4 of S. Jhanthanam (Jhanthanam *et al.*, 2019).

Example 7 From Theorem 7, if $a = 8,\ b = 9$ and $c = -9,$ then (34) becomes

$$t^3 y'''(t) + 8t^2 y''(t) + 9ty'(t) - 9y(t) = 0. \tag{45}$$

It follows form (35) and (36) that its solutions are

$$y(t) = \delta''(t) \quad \text{and} \quad y(t) = tH(t). \tag{46}$$

The solution is similar as example 4 of S. Jhanthanam (Jhanthanam *et al*., 2019).

Discussion

We study the generalized solutions of $t^2 y''(t) + aty'(t) + by(t) = 0,$ where a, b are integers and $t \in \mathbb{R}$, using Elzaki Transform technique. We find types of solutions depending on the values of a and b, also we have a distributional solution for $(k+2)(k+1) = (k+1)a - b,$ and a weak solution for $k - k^2 = ka + b,$ where $k \in \mathbb{N} \cup \{0\}$. Moreover, we study the generalized solutions of $t^3 y'''(t) + at^2 y''(t) + bty'(t) + cy(t) = 0,$ where a, b and c are integer and $t \in \mathbb{R}$. We find types of solutions depending on the values of a, b and c, also we have a distributional solution for $(k+3)(k+2)(k+1) = (k+2)(k+1)a - (k+1)b + c,$ and a weak solution for $k(k-2)(k-1) = -k(k-1)a - kb - c,$ where $k \in \mathbb{N} \cup \{0\}$. The solutions are in the form of a distributional solution $y(t) = \delta^{(k)}(t)$ and a weak solution $y(t) = \dfrac{t^k H(t)}{k!}$. The solution is similar as S. Jhanthanam (Jhanthanam *et al*., 2019), and H. Kim *et al*. (Kim *et al*., 2018) which gave the generalized solutions of the third-order Cauchy-Euler equation using Laplace Transform technique. In the future, we will devote our attention to the solutions of the nth-order Cauchy-Euler equation.

Conclusions

We used the Elzaki transform technique to find the generalized solutions of the second and third-order Cauchy-Euler equations of $t^2 y''(t) + aty'(t) + by(t) = 0,$ and $t^3 y'''(t) + at^2 y''(t) + bty'(t) + cy(t) = 0,$ where a, b and c are integers and $t \in \mathbb{R}$. Then, we applied the inverse Elzaki transform to the derived solutions. We found the conditions of a, b, and c for the case of a distributional solution and a weak solution. It is anticipated that our findings may encourage further research in this field.

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