วารสารวิทยาศาสตร์บูรพา ปีที่ 27 (ฉบับที่ 3) กันยายน – ธันวาคม พ.ศ. 2565
BURAPHA SCIENCE JOURNAL Volume 27 (No.3) September – December 2022 บทความวิจัย

เมทริกซ์เอกฐานที่เกี่ยวข้องกับจำนวน k -เพลล์และ k -เพลล์ลูคัส

Some Singular Matrices Related to k -Pell and k -Pell Lucas Numbers

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Received : 21 October 2021

Revised : 8 March 2022

Accepted : 22 March 2022

บทคัดย่อ

ในบทความนี้ ผู้วิจัยได้นำเสนอเมทริกซ์เอกฐานมิติขนาด 3×3 ที่สมาชิกทุกตัวของเมทริกซ์ยกกำลังที่ n^{th} เกี่ยวข้องกับจำนวน k -เพลล์และ k -เพลล์ลูคัส

คำสำคัญ : จำนวน k -เพลล์ ; จำนวน k -เพลล์ลูคัส ; เมทริกซ์

Abstract

In this paper, we will present new singular matrices of three-by-three dimensions, whose entries of any n^{th} powers of matrices are related to k -Pell and k -Pell Lucas numbers.

Keywords : k -Pell numbers ; k -Pell Lucas numbers ; Matrices

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Introduction

The sequence of real numbers has been widely studied in field of science over several years. Fibonacci $\{F_n\}_{n\in\mathbb{N}}$ and Lucas $\{L_n\}_{n\in\mathbb{N}}$ numbers are one of well-known numbers that have defined recursively by

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2, \tag{1}$$

and

$$L_n = L_{n-1} + L_{n-2}, \quad n \geq 2, \tag{2}$$

with initial conditions $F_0 = 0, F_1 = 1$ and $L_0 = 2, L_1 = 1$, respectively.

Many researchers have been studied these numbers in different ways such as their basic properties and generalizations. For any positive real number k, the k-Fibonacci numbers $\{F_{k,n}\}_{n\in\mathbb{N}}$ is one of generalization of Fibonacci numbers which was introduced by Falcon (Falcon & Plaza, 2007). It is defined recursively by

$$F_{k,n} = kF_{k,n-1} + F_{k,n-2}, \quad n \geq 2, \tag{3}$$

with initial conditions $F_{k,0} = 0, F_{k,1} = 1$. In particular case, when $k = 1$, it reduces to the classical Fibonacci numbers, and when $k = 2$, it reduces to the Pell numbers $\{P_n\}_{n\in\mathbb{N}}$ which is as important as Fibonacci numbers. It was introduced by Horadam (Horadam, 1971) that defined recursively by

$$P_n = 2P_{n-1} + P_{n-2}, \quad n \geq 2, \tag{4}$$

with initial conditions $P_0 = 0, P_1 = 1$. In addition, the Pell numbers associated to the Pell-Lucas numbers $\{Q_n\}_{n\in\mathbb{N}}$ because they have the same recurrence relation but the initial conditions are distinct. The Pell-Lucas numbers is defined recursively by

$$Q_n = 2Q_{n-1} + Q_{n-2}, \quad n \geq 2,, \tag{5}$$

with initial conditions $Q_0 = Q_1 = 2$.

Moreover, the basic properties, identities and generalizations of Pell and Pell-Lucas numbers have also been studied by several authors. For example, Catarino and Vasco considered generalizations of Pell and Pell-Lucas numbers, namely k-Pell numbers $\{P_{k,n}\}_{n\in\mathbb{N}}$ and k-Pell Lucas numbers $\{Q_{k,n}\}_{n\in\mathbb{N}}$ (Catarino, 2013; Catarino & Vasco, 2013). The k-Pell numbers is defined recursively by

$$P_{k,n} = 2P_{k,n-1} + kP_{k,n-2}, \quad n \geq 2, \tag{6}$$

with initial conditions $P_{k,0} = 0, P_{k,1} = 1$. The k-Pell Lucas numbers is defined recursively by

$$Q_{k,n} = 2Q_{k,n-1} + kQ_{k,n-2}, \quad n \geq 2, \tag{7}$$

with initial conditions $Q_{k,0} = Q_{k,1} = 2$.

Furthermore, the study of matrices which entries of n^{th} power matrices are elements of sequences of real numbers, have been appeared in many research. For example, Silvester (Silvester, 1979) studied on

Fibonacci numbers and investigated 2×2 matrix that obtained n^{th} power matrix as follows: $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ then $A^n \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix}$. In 2010, Koken and Bozkurt (Koken & Bozkurt, 2010) showed that if $S = \frac{1}{2}\begin{bmatrix} 1 & 5 \\ 1 & 1 \end{bmatrix}$ then $S^n = \frac{1}{2}\begin{bmatrix} L_n & 5F_n \\ F_n & L_n \end{bmatrix}$. Catarino (Catarino, 2013) defined 2×2 matrix as $T = \begin{bmatrix} 0 & 1 \\ k & 2 \end{bmatrix}$ then $T^n = \begin{bmatrix} kP_{k,n-1} & P_{k,n} \\ kP_{k,n} & P_{k,n+1} \end{bmatrix}$.

In 2018, Karakaya (Karakaya, 2018) developed a method for deriving 3×3 matrices whose powers are related to Fibonacci and Lucas numbers. After that, based on this approach, several authors have been developed the method to introduce 3×3 matrices whose elements of n^{th} power matrices related to different types of sequences such as Cerda-Morales, Koken and Petik et al. (Cerda-Morales, 2019; Koken, 2020; Petik et al, 2021)

In this paper, we will derive new 3×3 singular matrices through entries of n^{th} powers of matrices whose are related to the k-Pell and k-Pell Lucas numbers. The organization of this paper is as follows. We discuss the important properties and relation between k-Pell and k-Pell Lucas numbers in Section 2. The method for deriving matrices whose powers are related to the k-Pell and k-Pell Lucas numbers is introduced. Some special 3×3 nonsingular matrices are presented in Section 3. Finally, a discussion and a conclusion are provided in Section 4 and Section 5, respectively.

Methods

In this section, we consider k-Pell and k-Pell Lucas numbers which are defined in (6) and (7), respectively. For a present study, we are interested in the important properties of these numbers and their relations as follows:

Proposition 1. According to Binet's formula (Catarino, 2013), the n^{th} of k-Pell and k-Pell Lucas numbers which are given respectively:

$$P_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2} \qquad (8)$$

and

$$Q_{k,n} = r_1^n + r_2^n \text{ ,} \qquad (9)$$

where $r_1 = 1 + \sqrt{1+k}$ and $r_2 = 1 - \sqrt{1+k}$ are the roots of the same characteristic equation which is $x^2 - 2x - k = 0$.

From Proposition 1, solving equations (8) and (9) express r_1^n and r_2^n in terms of sequences. Then, for any integer n, we have

$$r_1^n = \frac{1}{2}\left(Q_{k,n} + 2P_{k,n}\sqrt{1+k}\right) \tag{10}$$

and

$$r_2^n = \frac{1}{2}\left(Q_{k,n} - 2P_{k,n}\sqrt{1+k}\right). \tag{11}$$

Proposition 2. (Catarino, 2013; Vasco *et al.,* 2015) For any positive real number k and positive integer n, the relationships between k-Pell and k-Pell Lucas numbers as follows:

1. $2P_{k,n+1} - 2P_{k,n} = Q_{k,n}$
2. $2P_{k,n} + 2kP_{k,n-1} = Q_{k,n}$
3. $(2+2k)P_{k,n} = Q_{k,n} + kQ_{k,n-1}$

Note that: The Proposition 2 holds for Pell and Pell-Lucas numbers when $k = 1$.

Results

In this section, we introduce a method to find some special matrices whose powers are related to k-Pell and k-Pell Lucas numbers.

Suppose that $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is any 3×3 matrix with eigenvalues $\lambda_1 = 4r_1, \lambda_2 = 4r_2$ and $\lambda_3 = 0$ with corresponding eigenvectors $\overline{\mathrm{x}} = [x_1 \quad x_2 \quad x_3]^t, \overline{\mathrm{y}} = [y_1 \quad y_2 \quad y_3]^t$ and $\overline{\mathrm{z}} = [z_1 \quad z_2 \quad z_3]^t$, respectively. So, we have a system of linear equations such as $A\overline{\mathrm{x}} = \lambda_1\overline{\mathrm{x}},\ A\overline{\mathrm{y}} = \lambda_2\overline{\mathrm{y}}$ and $A\overline{\mathrm{z}} = \lambda_3\overline{\mathrm{z}}$.

The matrix A is diagonalizable since the eigenvalues are distinct. So, without loss of the generality, we can write $A = T\Lambda T^{-1}$ where Λ is diagonal matrix with entries λ_1, λ_2 and λ_3. A matrix T is invertible matrix with column vectors $\overline{\mathrm{x}}, \overline{\mathrm{y}}$ and $\overline{\mathrm{z}}$. Thus, taking powers of diagonalizable matrices, we obtain $A^n = T\Lambda^n T^{-1}$;

$$A^n = T\begin{bmatrix} 4^n r_1^n & 0 & 0 \\ 0 & 4^n r_2^n & 0 \\ 0 & 0 & 0 \end{bmatrix} T^{-1}. \quad (12)$$

By (10) and (11), it follows that;

$$A^n = 2^{2n-1}\left(Q_{k,n} T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} T^{-1} - Q_{k,n} T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} T^{-1} + 2\sqrt{1+k}\, P_{k,n} T \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} T^{-1} \right).$$

Thus, we have formula of the power matrix A^n which related to k -Pell and k -Pell Lucas numbers as

$$A^n = 2^{2n-1}\left(Q_{k,n}\left(I - D \right) + 2\sqrt{1+k}\, P_{k,n} E \right), \quad (13)$$

where $D = T\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} T^{-1}$ and $E = T\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} T^{-1}$.

Now, we can derive some special matrix A which occur for the special cases of the eigenvectors $\overline{\mathrm{x}}, \overline{\mathrm{y}}$ and $\overline{\mathrm{z}}$ and some identities associated with these.

Firstly, we choose eigenvectors $\overline{\mathrm{x}} = \begin{bmatrix} r_1 \\ -2r_2 \\ 1 \end{bmatrix}, \overline{\mathrm{y}} = \begin{bmatrix} r_2 \\ -2r_1 \\ 1 \end{bmatrix}$. By assumption, we have a system of linear equations which satisfy $A\overline{\mathrm{x}} = \lambda_1 \overline{\mathrm{x}}$ and $A\overline{\mathrm{y}} = \lambda_2 \overline{\mathrm{y}}$. It is given by

$$a r_1 - 2b r_2 + c = 4r_1^2 \quad (14)$$

$$a r_2 - 2b r_1 + c = 4r_2^2 \quad (15)$$

$$d r_1 - 2e r_2 + f = -8r_1 r_2 \quad (16)$$

$$d r_2 - 2e r_1 + f = -8r_1 r_2 \quad (17)$$

$$g r_1 - 2h r_2 + i = 4r_1 \quad (18)$$

$$g r_2 - 2h r_1 + i = 4r_2 \quad (19)$$

From these equations, it is equivalent to

$$a - 2b + c = 8 + 4k, \qquad a + 2b = 8, \quad (20)$$

$$d - 2e + f = 8k, \qquad d + 2e = 0, \quad (21)$$

$$g - 2h + i = 4, \qquad g + 2h = 4. \quad (22)$$

Let us consider the different eigenvector $\overline{z}$ for eigenvectors $\overline{x}$ and $\overline{y}$ which are given above. If we choose $\overline{z} = \begin{bmatrix} -l \\ -2l \\ l \end{bmatrix}$ where $l \neq 0$ then the matrix D and E are obtained as

$$D = \frac{1}{4}\begin{bmatrix} 2 & -1 & -4 \\ 4 & -2 & -8 \\ -2 & 1 & 4 \end{bmatrix} \qquad (23)$$

and

$$E = \frac{1}{4\sqrt{1+k}}\begin{bmatrix} 2(1+k) & 1-k & 4 \\ 4(1+k) & -2(3+k) & -8 \\ 0 & 2 & 4 \end{bmatrix}. \qquad (24)$$

In addition, the system of equations which satisfy the systems $A\overline{z} = \lambda_3 \overline{z}$ given by

$$-a - 2b + c = 0 \qquad (25)$$

$$-d - 2e + f = 0 \qquad (26)$$

$$-g - 2h + i = 0\,. \qquad (27)$$

Solving system of linear equations $(20)-(22)$ and $(25)-(27)$, we obtain

$$A = \begin{bmatrix} 2(2+k) & 2-k & 8 \\ 4k & -2k & 0 \\ 2 & 1 & 4 \end{bmatrix}. \qquad (28)$$

Considering the formula given in (13) and substituting the matrices D and E, we get

$$A^n = 2^{2n-1}\left(Q_{k,n}(I-D) + 2\sqrt{1+k}\,P_{k,n}E\right)$$

$$= 2^{2n-1}\left(Q_{k,n}\left(I - \frac{1}{4}\begin{bmatrix} 2 & -1 & -4 \\ 4 & -2 & -8 \\ -2 & 1 & 4 \end{bmatrix}\right) + 2\sqrt{1+k}\,P_{k,n}\left(\frac{1}{4\sqrt{1+k}}\begin{bmatrix} 2(1+k) & 1-k & 4 \\ 4(1+k) & -2(3+k) & -8 \\ 0 & 2 & 4 \end{bmatrix}\right)\right)$$

$$= 2^{2n-3}\begin{bmatrix} 2\left(2(1+k)P_{k,n} + Q_{k,n}\right) & -2(-1+k)P_{k,n} + Q_{k,n} & 4\left(2P_{k,n} + Q_{k,n}\right) \\ 8(1+k)P_{k,n} - 4Q_{k,n} & -4(3+k)P_{k,n} + 6Q_{k,n} & 8\left(-2P_{k,n} + Q_{k,n}\right) \\ 2Q_{k,n} & 4P_{k,n} - Q_{k.n} & 8P_{k,n} \end{bmatrix}.$$

By using the identities given in Proposition 2, we can rewrite the entries of matrix. Thus, we get

$$A^n = 2^{2n-2}\begin{bmatrix} Q_{k,n+1} & P_{k,n+1} - kP_{k,n} & 4P_{k,n+1} \\ 2kQ_{k,n-1} & 2k\left(3P_{k,n-1} - P_{k,n}\right) & 8kP_{k,n-1} \\ Q_{k,n} & P_{k,n} - kP_{k,n-1} & 4P_{k,n} \end{bmatrix}. \tag{29}$$

For a positive integer $n \geq 1$. Thus, we have been proved the following theorem.

Theorem 1. *If* $A = \begin{bmatrix} 2(2+k) & 2-k & 8 \\ 4k & -2k & 0 \\ 2 & 1 & 4 \end{bmatrix}$*, then*

$$A^n = 2^{2n-2}\begin{bmatrix} Q_{k,n+1} & P_{k,n+1} - kP_{k,n} & 4P_{k,n+1} \\ 2kQ_{k,n-1} & 2k\left(3P_{k,n-1} - P_{k,n}\right) & 8kP_{k,n-1} \\ Q_{k,n} & P_{k,n} - kP_{k,n-1} & 4P_{k,n} \end{bmatrix}, \tag{30}$$

for a positive integer $n \geq 1$.

Let us consider the different eigenvectors

$$\overline{\mathrm{x}} \in \left\{\begin{bmatrix} r_1 \\ -2r_2 \\ 1 \end{bmatrix}, \begin{bmatrix} r_1 \\ 2r_2 \\ -1 \end{bmatrix}, \begin{bmatrix} -r_1 \\ 2r_2 \\ 1 \end{bmatrix}\right\},\ \overline{\mathrm{y}} \in \left\{\begin{bmatrix} r_2 \\ -2r_1 \\ 1 \end{bmatrix}, \begin{bmatrix} r_2 \\ 2r_1 \\ -1 \end{bmatrix}, \begin{bmatrix} -r_2 \\ 2r_1 \\ 1 \end{bmatrix}\right\} \text{ and } \overline{\mathrm{z}} \in \left\{\begin{bmatrix} -l \\ -2l \\ l \end{bmatrix}, \begin{bmatrix} l \\ -2l \\ -l \end{bmatrix}, \begin{bmatrix} l \\ 2l \\ l \end{bmatrix}\right\},$$

where $l \neq 0$. We have possibilities of the eigenvectors $\overline{\mathrm{x}}, \overline{\mathrm{y}}$ and $\overline{\mathrm{z}}$ corresponding to distinct eigenvalues λ_1, λ_2 and λ_3 that are linearly independent, then we receive the different matrices A and entries of A^n related to k-Pell and k-Pell Lucas numbers where k is a positive real number and a positive integer $n \geq 1$. The proof of Theorem 2 - 5, which is similarly to theorem 1, are omitted.

Theorem 2. *For eigenvalues and eigenvectors* $\left(\lambda_1, \begin{bmatrix} r_1 \\ -2r_2 \\ 1 \end{bmatrix}\right), \left(\lambda_2, \begin{bmatrix} r_2 \\ -2r_1 \\ 1 \end{bmatrix}\right)$*, we will have:*

1. If $\left(\lambda_3, \begin{bmatrix} l \\ -2l \\ -l \end{bmatrix}\right)$*, then the matrix* $A = \begin{bmatrix} 6+k & \frac{2-k}{2} & 2(2+k) \\ 2k & -k & 4k \\ 3 & \frac{1}{2} & 2 \end{bmatrix}$ *and*

$$A^n = 2^{2n-3}\begin{bmatrix} 2\left(3P_{k,n+1}+kP_{k,n}\right) & P_{k,n+1}-kP_{k,n} & 2Q_{k,n+1} \\ 4k\left(P_{k,n}+P_{k,n-1}\right) & 2k\left(3P_{k,n-1}-P_{k,n}\right) & 4kQ_{k,n-1} \\ 2\left(P_{k,n-1}+P_{k,n}\right) & P_{k,n}-kP_{k,n-1} & 2Q_{k,n} \end{bmatrix}. \quad (31)$$

2. If $\left(\lambda_3, \begin{bmatrix} l \\ 2l \\ l \end{bmatrix}\right)$, then the matrix $A = \begin{bmatrix} 2(6+k) & -2-k & -8 \\ 4k & -2k & 0 \\ 6 & -1 & -4 \end{bmatrix}$ and

$$A^n = 2^{2n-3}\begin{bmatrix} 4\left(3P_{k,n+1}+kP_{k,n}\right) & -Q_{k,n+1} & -8P_{k,n+1} \\ 8k\left(P_{k,n}+P_{k,n-1}\right) & -2kQ_{k,n-1} & -16kP_{k,n-1} \\ 4\left(P_{k,n+1}+P_{k,n}\right) & -Q_{k,n} & -8P_{k,n} \end{bmatrix}. \quad (32)$$

Theorem 3. *For eigenvalues and eigenvectors* $\left(\lambda_1, \begin{bmatrix} r_1 \\ 2r_2 \\ -1 \end{bmatrix}\right), \left(\lambda_2, \begin{bmatrix} r_2 \\ 2r_1 \\ -1 \end{bmatrix}\right)$, *we will have:*

1. If $\left(\lambda_3, \begin{bmatrix} -l \\ 2l \\ -l \end{bmatrix}\right)$, then the matrix $A = \begin{bmatrix} 2(2+k) & -2+k & -8 \\ -4k & -2k & 0 \\ -2 & 1 & 4 \end{bmatrix}$ and

$$A^n = 2^{2n-2}\begin{bmatrix} Q_{k,n+1} & kP_{k,n}-P_{k,n+1} & -4P_{k,n+1} \\ -2kQ_{k,n-1} & -2k\left(P_{k,n}-3P_{k,n-1}\right) & 8kP_{k,n-1} \\ -Q_{k,n} & P_{k,n}-kP_{k,n-1} & 4P_{k,n} \end{bmatrix}. \quad (33)$$

2. If $\left(\lambda_3, \begin{bmatrix} l \\ -2l \\ -l \end{bmatrix}\right)$, then the matrix $A = \begin{bmatrix} 2(6+k) & 2+k & 8 \\ -4k & -2k & 0 \\ -6 & -1 & -4 \end{bmatrix}$ and

$$A^n = 2^{2n-3}\begin{bmatrix} 4\left(3P_{k,n+1}+kP_{k,n}\right) & Q_{k,n+1} & 8P_{k,n+1} \\ -8k\left(P_{k,n}+P_{k,n-1}\right) & -2kQ_{k,n-1} & -16kP_{k,n-1} \\ -4\left(P_{k,n+1}+P_{k,n}\right) & -Q_{k,n} & -8P_{k,n} \end{bmatrix}. \quad (34)$$

Theorem 4. *For eigenvalues and eigenvectors* $\left(\lambda_1, \begin{bmatrix} r_1 \\ 2r_2 \\ -1 \end{bmatrix}\right), \left(\lambda_2, \begin{bmatrix} r_2 \\ 2r_1 \\ -1 \end{bmatrix}\right)$, *we will have:*

1. If $\left(\lambda_3, \begin{bmatrix} -l \\ 2l \\ -l \end{bmatrix}\right)$, *then the matrix* $A = \begin{bmatrix} 6+k & \frac{2-k}{2} & -2(2+k) \\ 2k & -k & -4k \\ -3 & -\frac{1}{2} & 2 \end{bmatrix}$ *and*

$$A^n = 2^{2n-3}\begin{bmatrix} 2(3P_{k,n+1}+kP_{k,n}) & P_{k,n+1}-kP_{k,n} & -2Q_{k,n+1} \\ 4k(P_{k,n}+P_{k,n-1}) & -2k(P_{k,n}-3P_{k,n-1}) & -4kQ_{k,n-1} \\ -2(P_{k,n+1}+P_{k,n}) & kP_{k,n-1}-P_{k,n} & 2Q_{k,n} \end{bmatrix}, . \quad (35)$$

2. If $\left(\lambda_3, \begin{bmatrix} l \\ 2l \\ l \end{bmatrix}\right)$, *then the matrix* $A = \begin{bmatrix} 2(2+k) & 2-k & -8 \\ 4k & -2k & 0 \\ -2 & -1 & 4 \end{bmatrix}$ *and*

$$A^n = 2^{2n-2}\begin{bmatrix} Q_{k,n+1} & P_{k,n+1}-kP_{k,n} & -4P_{k,n+1} \\ 2kQ_{k,n-1} & -2k(P_{k,n}-3P_{k,n-1}) & -8kP_{k,n-1} \\ -Q_{k,n} & kP_{k,n-1}-P_{k,n} & 4P_{k,n} \end{bmatrix}. \quad (36)$$

Theorem 5. *For eigenvalues and eigenvectors* $\left(\lambda_1, \begin{bmatrix} -r_1 \\ 2r_2 \\ 1 \end{bmatrix}\right), \left(\lambda_2, \begin{bmatrix} -r_2 \\ 2r_1 \\ 1 \end{bmatrix}\right)$, *we will have:*

1. If $\left(\lambda_3, \begin{bmatrix} -l \\ -2l \\ l \end{bmatrix}\right)$, *then the matrix* $A = \begin{bmatrix} 2(6+k) & -2-k & 8 \\ 4k & -2k & 0 \\ -6 & 1 & -4 \end{bmatrix}$ *and*

$$A^n = 2^{2n-3}\begin{bmatrix} 4(3P_{k,n+1}+kP_{k,n}) & -Q_{k,n+1} & 8P_{k,n+1} \\ 8k(P_{k,n}+P_{k,n-1}) & -2kQ_{k,n-1} & 16kP_{k,n-1} \\ -4(P_{k,n+1}+P_{k,n}) & Q_{k,n} & -8P_{k,n} \end{bmatrix}. \quad (37)$$

Corollary 6. For any positive integer n, the following equalities are satisfied.

1. $\begin{bmatrix} 6 & 1 & 8 \\ 4 & -2 & 0 \\ 2 & 1 & 4 \end{bmatrix}^n = 2^{2n-3}\begin{bmatrix} 2Q_{n+1} & Q_n & 8P_{n+1} \\ 4Q_{n-1} & 2Q_{n-2} & 16P_{n-1} \\ 4Q_n & Q_{n-1} & 8P_n \end{bmatrix}$

2. $\begin{bmatrix} 7 & \frac{1}{2} & 6 \\ 2 & -1 & 4 \\ 3 & \frac{1}{2} & 2 \end{bmatrix}^n = 2^{2n-4}\begin{bmatrix} 2Q_{n+2} & Q_n & 4Q_{n+1} \\ 4Q_n & 2Q_{n-2} & 8Q_{n-1} \\ Q_{n+1} & Q_{n-1} & 4Q_n \end{bmatrix}$

3. $\begin{bmatrix} 14 & -3 & -8 \\ 4 & -2 & 0 \\ 6 & -1 & -4 \end{bmatrix}^n = 2^{2n-3}\begin{bmatrix} 2Q_{n+2} & -Q_{n+1} & -8P_{n+1} \\ 4Q_n & -2Q_{n-1} & -16P_{n-1} \\ 2Q_{n+1} & -Q_n & -8P_n \end{bmatrix}$

4. $\begin{bmatrix} 6 & -1 & -8 \\ -4 & -2 & 0 \\ -2 & 1 & 4 \end{bmatrix}^n = 2^{2n-3}\begin{bmatrix} 2Q_{n+1} & -Q_n & -8P_{n+1} \\ -4Q_{n-1} & 2Q_{n-2} & 16P_{n-1} \\ -2Q_n & Q_{n-1} & 8P_n \end{bmatrix}$

5. $\begin{bmatrix} 14 & 3 & 8 \\ -4 & -2 & 0 \\ -6 & -1 & -4 \end{bmatrix}^n = 2^{2n-3}\begin{bmatrix} 2Q_{n+2} & Q_{n+1} & 8P_{n+1} \\ -4Q_n & -2Q_{n-1} & -16P_{n-1} \\ -2Q_{n+1} & -Q_n & -8P_n \end{bmatrix}$

6. $\begin{bmatrix} 7 & \frac{1}{2} & -6 \\ 2 & -1 & -4 \\ -3 & -\frac{1}{2} & 2 \end{bmatrix}^n = 2^{2n-4}\begin{bmatrix} 2Q_{n+2} & Q_n & -4Q_{n+1} \\ 4Q_n & 2Q_{n-2} & -8Q_{n-1} \\ -2Q_{n+1} & -Q_{n-1} & 4Q_n \end{bmatrix}$

7. $\begin{bmatrix} 6 & 1 & -8 \\ 4 & -2 & 0 \\ -2 & -1 & 4 \end{bmatrix}^n = 2^{2n-3}\begin{bmatrix} 2Q_{n+1} & Q_n & -8P_{n+1} \\ 4Q_{n-1} & 2Q_{n-2} & -16P_{n-1} \\ -2Q_n & -Q_{n-1} & 8P_n \end{bmatrix}$

8. $\begin{bmatrix} 14 & -3 & 8 \\ 4 & -2 & 0 \\ -6 & 1 & -4 \end{bmatrix}^n = 2^{2n-3}\begin{bmatrix} 2Q_{n+2} & -Q_{n+1} & 8P_{n+1} \\ 4Q_n & -2Q_{n-1} & 16P_{n-1} \\ -2Q_{n+1} & Q_n & -8P_n \end{bmatrix}$

Proof. In Theorem 1 - 5, by taking $k = 1$ and using Proposition 2, the entries of power matrices are reduced to Pell and Pell-Lucas numbers. . □

Discussion

In discussion, we studied k -Pell and k -Pell Lucas numbers and the matrix method. In the results, we show the method to express the n^{th} powers of three-by-three matrices such that entries related to k -Pell and k -Pell Lucas numbers. Finally, the eight n^{th} powers matrices by choosing only one eigenvector from $\overline{x}, \overline{y}$ and $\overline{z}$ are presented.

Conclusion

In this paper, we obtained the matrices which entries of power matrix are k -Pell and k -Pell Lucas numbers. Moreover, if $k = 1$, some matrices which entries of power matrix are Pell and Pell Lucas numbers are given.

For the future study, we intend to find some matrices that are invertible and contain entries of power matrices in other sequences.

Acknowledgment

We would like to thank the referees for their useful comments and suggestions on the manuscript. This work is supported by Research and Development Institute, Sakon Nakhon Rajabhat University under Grant No.4/2563

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