วารสารวิทยาศาสตร์บูรพา ปีที่ 27 (ฉบับที่ 1) มกราคม – เมษายน พ.ศ. 2565
BURAPHA SCIENCE JOURNAL Volume 27 (No.1) January – April 2022 บทความวิจัย

เอกลักษณ์เบื้องต้นของจำนวนโมดิฟายด์ (s,t) จาคอปทอล และ จำนวนโมดิฟายด์ (s,t) จาคอปทอล-ลูคัสโดยเมทริกซ์

Some Identities of the Modified (s,t) Jacobsthal and Modified (s,t) Jacobsthal – Lucas Numbers by the Matrix Method

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Received : 22 February 2021

Revised : 18 May 2021

Accepted : 16 June 2021

บทคัดย่อ

ในงานวิจัยนี้เราได้ศึกษาจำนวนโมดิฟายด์ (s,t) จาคอปทอล และ จำนวนโมดิฟายด์ (s,t) จาคอปทอล-ลูคัส และนิยามเมทริกซ์มิติ 2 x 2 A B และ W ซึ่งสอดคล้องกับความสัมพันธ์ $A^2=(s-t)A+stI$ $B^2=(s-t)B+stI$ และ $W^2=(s+t)^2 I$ พร้อมทั้งพิสูจน์เอกลักษณ์เบื้องต้นของจำนวนโมดิฟายด์ (s,t) จาคอปทอล และ จำนวนโมดิฟายด์ (s,t) จาคอปทอล-ลูคัส เอกลักษณ์เบื้องต้นของความสัมพันธ์ระหว่างจำนวนโมดิฟายด์ (s,t) จาคอปทอล และ จำนวนโมดิฟายด์ (s,t) จาคอปทอล-ลูคัส และสูตรผลรวมเบื้องต้นสำหรับจำนวนโมดิฟายด์ (s,t) จาคอปทอล และ จำนวนโมดิฟายด์ (s,t) จาคอปทอล-ลูคัสโดยใช้เมทริกซ์

คำสำคัญ: จำนวนโมดิฟายด์ (s,t) จาคอปทอล; จำนวนโมดิฟายด์ (s,t) จาคอปทอล-ลูคัส; วิธีเมทริกซ์; สูตรไบเนต

Abstract

In this paper, we study the modified (s,t) Jacobsthal and modified (s,t) Jacobsthal – Lucas numbers, and we define the 2 x 2 matrices A, B, W, which satisfy the relation $A^2=(s-t)A+stI$, $B^2=(s-t)B+stI$, and $W^2=(s+t)^2 I$. Moreover, we prove some identities of modified (s,t) Jacobsthal and modified (s,t) Jacobsthal – Lucas numbers, some of the relation between modified (s,t) Jacobsthal and modified (s,t) Jacobsthal – Lucas numbers, and some sum formulas for modified (s,t) Jacobsthal and modified (s,t) Jacobsthal – Lucas numbers by using these matrices.

Keywords: modified (s,t) Jacobsthal number; modified (s,t) Jacobsthal – Lucas number; matrix method; Binet's formulas

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Introduction

The Fibonacci sequence $\{F_n\}$ and Lucas sequence $\{L_n\}$ are the two most well-known sequences, and these sequences are defined respectively by the recurrence relations $F_n = F_{n-1} + F_{n-2}$ and $L_n = L_{n-1} + L_{n-2}$, for $n \ge 3$, with initial conditions $F_1 = 1$, $F_2 = 1$, $L_1 = 1$, and $L_2 = 3$. (Horadam, A. F., 1961), (Clarke, J.H. & Shannon, A.G., 1985).

In 1996, Alwyn F. Horadam studied the Jacohsthal sequence $\{U_n\}$ and Jacobsthal – Lucas sequence $\{V_n\}$. For $n \ge 0$, these sequences are defined respectively by the recurrence relations $U_{n+2} = U_{n+1} + 2U_n$ and $V_{n+2} = V_{n+1} + 2V_n$, with initial conditions $U_0 = 0$, $U_1 = 1$, $V_0 = 2$, and $V_1 = 1$.

In 2008, Fikri Koken and Durmus Bozkurt studied the H-matrix and M-matrix. These matrices are defined respectively by $H = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$ and $M = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$. Also, they obtained some identities of the Jacohsthal numbers U_n and Jacobsthal – Lucas numbers V_n using these matrices and elementary matrix algebra.

In 2014, Julius Fergy T. Rabago studied the modified (s,t) Jacobsthal sequence $\{J_n^{s,t}\}$ and modified (s,t) Jacobsthal – Lucas sequence $\{j_n^{s,t}\}$. For $n \ge 1$, these sequences are defined respectively by the recurrence relations

$$J_{n+1}^{s,t} = (s-t)J_n^{s,t} + stJ_{n-1}^{s,t}, \tag{1}$$

$$j_{n+1}^{s,t} = (s-t)j_n^{s,t} + stj_{n-1}^{s,t}, \tag{2}$$

with initial conditions $J_0^{s,t} = 0$, $J_1^{s,t} = 1$, $j_0^{s,t} = 2$, and $j_1^{s,t} = s-t$. The first few terms of the modified (s,t) Jacobsthal numbers $J_n^{s,t}$ and modified (s,t) Jacobsthal – Lucas numbers $j_n^{s,t}$, which are respectively created via the recurrence relations in (1) and (2) as follows :

Table 1 The first few terms of $J_n^{s,t}$ and $j_n^{s,t}$, for $n = 0, 1, 2, 3$.

n :	0	1	2	3
$J_n^{s,t}$:	0	1	$s-t$	$s^2 - st + t^2$
$j_n^{s,t}$:	2	$s-t$	$s^2 + t^2$	$s^3 - t^3$

In the particular case of (1) and (2) are: if $s = \frac{1-\sqrt{5}}{2}$, $t = -\frac{1+\sqrt{5}}{2}$ and $s = \frac{1+\sqrt{5}}{2}$, $t = -\frac{1-\sqrt{5}}{2}$ then the classical Fibonacci and Lucas sequences are obtained, and if $s = -1$, $t = -2$ and $s = 2$, $t = 1$ then the classical Jacobsthal and Jacobsthal – Lucas sequences are obtained. Also, he obtained some identities of the modified (s,t) Jacobsthal numbers and modified (s,t) Jacobsthal – Lucas numbers using matrix algebra.

In 2015, Julius Fergy T. Rabago studied Binet's formulas of the recurrence relations (1) and (2) as follows: For a natural number n, their well-known formulas are defined respectively by

$$J_n^{s,t} = \frac{s^n - (-t)^n}{s+t} \text{ and } j_n^{s,t} = s^n + (-t)^n, \tag{3}$$

where $x_{1,2} = s$ and $-t$ are roots of the characteristic equation $x^2 - (s-t)x - st = 0$, s, t are any real numbers and $s \neq -t$. Note that $x_1 + x_2 = s - t$, $x_1 - x_2 = s + t$, and $x_1 x_2 = -st$. For convenience throughout this paper, we will use the symbols J_n, j_n instead of $J_n^{s,t}$ and $j_n^{s,t}$, respectively.

Methods

In this section, firstly, we give definitions of the X-matrix and Y-matrix, which satisfy the relations $X^2 = (s-t)X + stI$ and $Y^2 = (s+t)^2 I$, respectively.

Definition 1 Let a, b, p, q, s, and t be real numbers such that $b, q \neq 0$ and $s \neq -t$. Then the $\boldsymbol{X}$ **-matrix** and $\boldsymbol{Y}$ **-matrix** can be written as

$$X = \begin{pmatrix} a & b \\ -\dfrac{-st - sa + ta + a^2}{b} & s - t - a \end{pmatrix}, \tag{4}$$

and

$$Y = \begin{pmatrix} p & q \\ \dfrac{(s+t)^2 - p^2}{q} & -p \end{pmatrix}. \tag{5}$$

Next, we find the nth power of the X-matrix in which the component matrix consists of J_n, as shown in the following lemma and theorem.

Lemma 2 For $n \geq 1$. Then the nth power of the X-matrix is given by

$$X^n = J_n X + stJ_{n-1} I. \tag{6}$$

Proof. We will prove this by mathematical induction that $X^n = J_n X + stJ_{n-1} I$ for $n \geq 1$.

It is not hard to see that $X = J_1 X + stJ_0 I$. Thus (6) holds for $n = 1$.

Assume that the result is true for the positive integer, $n = k$ then

$$X^k = J_k X + stJ_{k-1} I.$$

Next, we need to show that (6) also holds for $n = k+1$ by considering (1) and Definition 1 as follows:

$$\begin{aligned} X^{k+1} &= X^k X \\ &= \left(J_k X + stJ_{k-1} I\right) X \\ &= J_k X^2 + stJ_{k-1} X \\ &= J_k\left((s-t)X + stI\right) + stJ_{k-1} X \end{aligned}$$

$$
\begin{aligned}
&=(s-t)J_kX+stJ_kI+stJ_{k-1}X\\
&=(s-t)J_kX+stJ_{k-1}X+stJ_kI\\
&=\left((s-t)J_k+stJ_{k-1}\right)X+stJ_kI\\
&=J_{k+1}X+stJ_kI\ .
\end{aligned}
$$

Therefore $n=k+1$ is true, and this completes the proof.

Theorem 3 For $n\geq 1$ and $b\neq 0$, we have

$$X^n=\begin{pmatrix} aJ_n+stJ_{n-1} & bJ_n \\ -\dfrac{-st-sa+ta+a^2}{b}J_n & (s-t-a)J_n+stJ_{n-1}\end{pmatrix}. \tag{7}$$

Proof. It is immediately proven by (6).

Now, we define the 2x2 matrices A, B, and W. These matrices satisfy the relations $A^2=(s-t)A+stI$, $B^2=(s-t)B+stI$, and $W^2=(s+t)^2I$ in which the component of each matrix consists of J_1, J_2, j_0, and j_1 as follows :

Definition 4 Let s and t are a real number such that $s\neq -t$. Then the $\boldsymbol{A}$**-matrix**, $\boldsymbol{B}$**-matrix**, and $\boldsymbol{W}$**-matrix** are defined respectively by

$$A=\begin{pmatrix} s-t & st \\ 1 & 0\end{pmatrix}, \tag{8}$$

$$B=\begin{pmatrix} \dfrac{s-t}{2} & \dfrac{(s+t)^2}{2} \\ \dfrac{1}{2} & \dfrac{s-t}{2}\end{pmatrix}, \tag{9}$$

and

$$W=\begin{pmatrix} s-t & 2st \\ 2 & -(s-t)\end{pmatrix}. \tag{10}$$

For some particular values of a and b in (4), it is obvious the following results hold.

- If $a=J_2=s-t$ and $b=stJ_1=st$, then (8) is obtained.
- If $a=\frac{j_1}{2}=\frac{s-t}{2}$ and $b=\frac{(s+t)^2}{2}J_1=\frac{(s+t)^2}{2}$, then (9) is obtained.

Also, for some particular values of a and b in (5), it is obvious the following results hold.

- If $p=j_1=s-t$ and $q=stj_0=2st$, then (10) is obtained.

After that, we find the n[th] power of the A-matrix and B-matrix, which corresponds to the following theorem.

Lemma 5 For $n \geq 1$. Then the n^th power of the A-matrix and B-matrix are given respectively by

(i) $A^n = J_n A + stJ_{n-1} I$,

(ii) $B^n = J_n B + stJ_{n-1} I$.

Proof. The proofs of (i) and (ii) are similar to (6) by using (1) and Definition 4.

Theorem 6 For $n \geq 1$, we have

(i) $A^n = \begin{pmatrix} J_{n+1} & stJ_n \\ J_n & stJ_{n-1} \end{pmatrix}$,

(ii) $B^n = \begin{pmatrix} \frac{1}{2} j_n & \frac{(s+t)^2}{2} J_n \\ \frac{1}{2} J_n & \frac{1}{2} j_n \end{pmatrix}$.

Proof. Taking $a = s - t$ and $b = st$ in (7), then we have

$$A^n = \begin{pmatrix} (s-t) J_n + stJ_{n-1} & stJ_n \\ J_n & stJ_{n-1} \end{pmatrix}.$$

By (1), we have

$$A^n = \begin{pmatrix} J_{n+1} & stJ_n \\ J_n & stJ_{n-1} \end{pmatrix}.$$

The proof of (ii) is similar to (i).

Furthermore, we find the n^th power of the A-matrix, which multiplies the W-matrix, as shown in the following theorem.

Theorem 7 For $n \geq 1$, we get

$$A^n W = WA^n = j_n A + stj_{n-1} I = \begin{pmatrix} j_{n+1} & stj_n \\ j_n & stj_{n-1} \end{pmatrix}. \quad (11)$$

Proof. The proof of (11) is similar to (6) by using (2) and Definition 4.

Results

In this section, we first find some identities of J_n and j_n. We also find some identities of the relations between J_n and j_n, as shown in the following lemma.

Lemma 8 For $n, r \geq 1$. Then

(i) $(s-t) J_n + 2stJ_{n-1} = j_n$,

(ii) $2J_{n+1} - (s-t) J_n = j_n$,

(iii) $J_{n+1} + stJ_{n-1} = j_n$,

(iv) $(s^2 + t^2) J_n + st(s-t) J_{n-1} = j_{n+1}$,

(v) $j_{n+1}+stj_{n-1}=(s+t)^2 J_n$,

(vi) $(s-t)j_{n+r+1}+2stj_{n+r}=(s+t)j_{n+r+1}$,

(vii) $2j_{n+r+1}-(s-t)j_{n+r}=(s+t)^2 J_{n+r}$.

Proof. By the Binet's formulas in (3), we have

$$(s-t)J_n+2stJ_{n-1}=(s-t)\left(\frac{s^n-(-t)^n}{s+t}\right)+2st\left(\frac{s^{n-1}-(-t)^{n-1}}{s+t}\right)=s^n+(-t)^n=j_n .$$

The proofs of (ii), (iii), (iv), (v), (vi), and (vii) are similar to (i).

After that, we find some identities of J_n and j_n. We also find some identities of the relations between J_n and j_n by using A^n, B^n, A^nW, and WA^n, as follows:

Lemma 9 For $n, r \geq 1$ and $n-r \geq 0$. Then

(i) $det(A^n)=(-1)^n(st)^n$,

(ii) $J_{n-1}J_{n+1}-J_n^2=(-1)^n(st)^{n-1}$,

(iii) $J_{n+r}=stJ_{n-1}J_r+J_nJ_{r+1}$,

(iv) $(-1)^r(st)^{r-1}J_{n-r}=J_nJ_{r-1}-J_{n-1}J_r$.

Proof. By $\det(A)=-st$, we have

$$det(A^n)=(detA)^n=(-1)^n(st)^n . \tag{12}$$

It follows by Theorem 6 (i) that

$$det(A^n)=stJ_{n-1}J_{n+1}-stJ_n^2 . \tag{13}$$

By using (12) and (13), we obtain

$$J_{n-1}J_{n+1}-J_n^2=(-1)^n(st)^{n-1} .$$

Since $A^{n+r}=A^nA^r$ then

$$\begin{pmatrix} J_{n+r+1} & stJ_{n+r} \\ J_{n+r} & stJ_{n+r-1}\end{pmatrix}=\begin{pmatrix} stJ_nJ_r+J_{n+1}J_{r+1} & s^2t^2J_nJ_{r-1}+stJ_{n+1}J_r \\ stJ_{n-1}J_r+J_nJ_{r+1} & s^2t^2J_{n-1}J_{r-1}+stJ_nJ_r\end{pmatrix}.$$

Note that

$$A^{-r}=\frac{1}{(-st)^r}\begin{pmatrix} stJ_{r-1} & -stJ_r \\ -J_r & J_{r+1}\end{pmatrix}.$$

Since $A^{n-r}=A^n(A^{-r})=A^n(A^r)^{-1}$ we obtain

$$\begin{pmatrix} J_{n-r+1} & stJ_{n-r} \\ J_{n-r} & stJ_{n-r-1}\end{pmatrix}=\frac{1}{(-1)^r(st)^{r-1}}\begin{pmatrix} J_{n+1}J_{r-1}-J_nJ_r & J_nJ_{r+1}-J_{n+1}J_r \\ J_nJ_{r-1}-J_{n-1}J_r & J_{n-1}J_{r+1}-J_nJ_r\end{pmatrix}.$$

Therefore, the identities (i), (ii), (iii), and (iv) are immediately seen.

Lemma 10 For $n \geq 0$. Then the following results hold.

(i) $det(B^n) = (-1)^n (st)^n$,

(ii) $det(B^n) = \frac{j_n^2}{4} - \frac{(s+t)^2 J_n^2}{4}$.

Proof. The proofs of (i) and (ii) are similar to Lemma 9 (i).

Lemma 11 For $n, r \geq 1$ and $n-r \geq 0$. Then the following results hold.

(i) $\det(WA^n) = (-1)^{n+1}(s+t)^2(st)^n$,

(ii) $j_{n+1}j_{n-1} - j_n^2 = (-1)^{n+1}(s+t)^2(st)^{n-1}$,

(iii) $j_{n+r} = stj_{n-1}J_r + j_n J_{r+1}$,

(iv) $(-1)^r (st)^{r-1} j_{n-r} = j_n J_{r-1} - j_{n-1}J_r$.

Proof. The proofs of (i), (ii), (iii), and (iv) are similar to Lemma 9 (i), (ii), (iii), and (iv).

Theorem 12 For $n, r \geq 0$ and $n-r \geq 0$, we have

$$J_{n+r} - (-st)^r J_{n-r} = j_n J_r . \tag{14}$$

Proof. It is known that

$$\begin{aligned} W^2A^{n+r} - (-st)^r W^2A^{n-r} &= WWA^nA^r - (-st)^r WWA^nA^{-r} \\ &= WA^nWA^r - (-st)^r WA^nWA^{-r} \\ &= (WA^n)(WA^r) - (-st)^r (WA^n)(WA^{-r}) \\ &= WA^n\left(WA^r - (-st)^r WA^{-r}\right) \\ &= WA^n\left(WA^r - (-st)^r W\left(A^r\right)^{-1}\right). \end{aligned}$$

Since matrix multiplication and matrix subtraction, we get

$$\begin{aligned} &W^2A^{n+r} - (-st)^r W^2A^{n-r} \\ &= W\left(WA^{n+r}\right) - (-st)^r W\left(WA^{n-r}\right) \\ &= \begin{pmatrix} s-t & 2st \\ 2 & -(s-t) \end{pmatrix}\begin{pmatrix} j_{n+r+1} & stj_{n+r} \\ j_{n+r} & stj_{n+r-1} \end{pmatrix} - (-st)^r \begin{pmatrix} s-t & 2st \\ 2 & -(s-t) \end{pmatrix}\begin{pmatrix} j_{n-r+1} & stj_{n-r} \\ j_{n-r} & stj_{n-r-1} \end{pmatrix} \\ &= \begin{pmatrix} (s-t)j_{n+r+1} + 2stj_{n+r} & st\left((s-t)j_{n+r} + 2stj_{n+r-1}\right) \\ 2j_{n+r+1} - (s-t)j_{n+r} & st\left(2j_{n+r} - (s-t)j_{n+r-1}\right) \end{pmatrix} - (-st)^r \begin{pmatrix} (s-t)j_{n-r+1} + 2stj_{n-r} & st\left((s-t)j_{n-r} + 2stj_{n-r-1}\right) \\ 2j_{n-r+1} - (s-t)j_{n-r} & st\left(2j_{n-r} - (s-t)j_{n-r-1}\right) \end{pmatrix}. \end{aligned} \tag{15}$$

By Lemma 8 (vi) and (vii), we can write

$$\begin{pmatrix} (s-t)j_{n+r+1} + 2stj_{n+r} & st\left((s-t)j_{n+r} + 2stj_{n+r-1}\right) \\ 2j_{n+r+1} - (s-t)j_{n+r} & st\left(2j_{n+r} - (s-t)j_{n+r-1}\right) \end{pmatrix} - (-st)^r \begin{pmatrix} (s-t)j_{n-r+1} + 2stj_{n-r} & st\left((s-t)j_{n-r} + 2stj_{n-r-1}\right) \\ 2j_{n-r+1} - (s-t)j_{n-r} & st\left(2j_{n-r} - (s-t)j_{n-r-1}\right) \end{pmatrix}$$

$$=\begin{pmatrix}(s+t)j_{n+r+1} & st(s+t)j_{n+r}\\ (s+t)^2J_{n+r} & st(s+t)^2J_{n+r-1}\end{pmatrix}-(-st)^r\begin{pmatrix}(s+t)j_{n-r+1} & st(s+t)j_{n-r}\\ (s+t)^2J_{n-r} & st(s+t)^2J_{n-r-1}\end{pmatrix}$$

$$=\begin{pmatrix}(s+t)\left(j_{n+r+1}-(-st)^r j_{n-r+1}\right) & st(s+t)\left(j_{n+r}-(-st)^r j_{n-r}\right)\\ (s+t)^2\left(J_{n+r}-(-st)^r J_{n-r}\right) & st(s+t)^2\left(J_{n+r-1}-(-st)^r J_{n-r-1}\right)\end{pmatrix}. \tag{16}$$

By using (16) in (15), we get that

$$W^2A^{n+r}-(-st)^r W^2A^{n-r}=\begin{pmatrix}(s+t)\left(j_{n+r+1}-(-st)^r j_{n-r+1}\right) & st(s+t)\left(j_{n+r}-(-st)^r j_{n-r}\right)\\ (s+t)^2\left(J_{n+r}-(-st)^r J_{n-r}\right) & st(s+t)^2\left(J_{n+r-1}-(-st)^r J_{n-r-1}\right)\end{pmatrix}. \tag{17}$$

Since $(-st)^r \neq 0$ and the matrix multiplication, we obtain that

$$WA^n\left(WA^r-(-st)^r W\left(A^r\right)^{-1}\right)$$

$$=\begin{pmatrix}j_{n+1} & stj_n\\ j_n & stj_{n-1}\end{pmatrix}\left(\begin{pmatrix}j_{r+1} & stj_r\\ j_r & stj_{r-1}\end{pmatrix}-\begin{pmatrix}st\left(-2J_r+(s-t)J_{r-1}\right) & st\left(2J_{r+1}-(s-t)J_r\right)\\ (s-t)J_r+2stJ_{r-1} & -(s-t)J_{r+1}-2stJ_r\end{pmatrix}\right). \tag{18}$$

By Lemma 8 (i) and (ii), we have

$$\begin{pmatrix}st\left(-2J_r+(s-t)J_{r-1}\right) & st\left(2J_{r+1}-(s-t)J_r\right)\\ (s-t)J_r+2stJ_{r-1} & -(s-t)J_{r+1}-2stJ_r\end{pmatrix}=\begin{pmatrix}-stj_{r-1} & stj_r\\ j_r & -j_{r+1}\end{pmatrix}. \tag{19}$$

By using (19) in (18) and matrix subtraction, we get

$$WA^n\left(WA^r-(-st)^r W\left(A^r\right)^{-1}\right)=\begin{pmatrix}j_{n+1} & stj_n\\ j_n & stj_{n-1}\end{pmatrix}\left(\begin{pmatrix}j_{r+1} & stj_r\\ j_r & stj_{r-1}\end{pmatrix}-\begin{pmatrix}-stj_{r-1} & stj_r\\ j_r & -j_{r+1}\end{pmatrix}\right)$$

$$=\begin{pmatrix}j_{n+1} & stj_n\\ j_n & stj_{n-1}\end{pmatrix}\begin{pmatrix}j_{r+1}+stj_{r-1} & 0\\ 0 & j_{r+1}+stj_{r-1}\end{pmatrix}. \tag{20}$$

By Lemma 8 (v) in (20) and matrix multiplication, we get

$$WA^n\left(WA^r-(-st)^r W\left(A^r\right)^{-1}\right)=\begin{pmatrix}j_{n+1} & stj_n\\ j_n & stj_{n-1}\end{pmatrix}\begin{pmatrix}(s+t)^2J_r & 0\\ 0 & (s+t)^2J_r\end{pmatrix}$$

$$=\begin{pmatrix}(s+t)^2j_{n+1}J_r & st(s+t)^2j_nJ_r\\ (s+t)^2j_nJ_r & st(s+t)^2j_{n-1}J_r\end{pmatrix}. \tag{21}$$

On the other hand, using (17) and (21), we obtain

$$J_{n+r}-(-st)^r J_{n-r}=j_nJ_r\,.$$

Finally, we find some sum formulas of J_n and j_n by using A^n, B^n, A^nW, and WA^n as follows:

Theorem 13 For $n, r \geq 0$, we have

$$\sum_{i=0}^{r}J_{ni}=\frac{\left(1-J_{nr+n+1}\right)J_n-\left(1-J_{n+1}\right)J_{nr+n}}{1-j_n+(-1)^n(st)^n}\,. \tag{22}$$

Proof. It is known that $I-\left(A^n\right)^{r+1}=\left(I-A^n\right)\sum_{i=0}^{r}\left(A^n\right)^i=\left(I-A^n\right)\sum_{i=0}^{r}A^{ni}$.

By Lemma 8 (iii) and Lemma 9 (ii), we get

$$det\left(I-A^n\right)=1-stJ_{n-1}-stJ_n^2-J_{n+1}+stJ_{n-1}J_{n+1}=1-j_n+(-1)^n(st)^n.$$

Since $\det\left(I-A^n\right)\neq 0$ we obtain

$$\left(I-A^n\right)^{-1}\left(I-\left(A^n\right)^{r+1}\right)=\sum_{i=0}^{r}A^{ni}=\begin{pmatrix}\sum_{i=0}^{r}J_{ni+1} & st\sum_{i=0}^{r}J_{ni}\\ \sum_{i=0}^{r}J_{ni} & st\sum_{i=0}^{r}J_{ni-1}\end{pmatrix}. \tag{23}$$

Since $\left(I-A^n\right)^{-1}=\dfrac{1}{1-j_n+(-1)^n(st)^n}\begin{pmatrix}1-stJ_{n-1} & stJ_n\\ J_n & 1-J_{n+1}\end{pmatrix}$ we have

$$\begin{aligned}&\left(I-A^n\right)^{-1}\left(I-A^{nr+n}\right)\\ &=\frac{1}{1-j_n+(-1)^n(st)^n}\begin{pmatrix}1-stJ_{n-1} & stJ_n\\ J_n & 1-J_{n+1}\end{pmatrix}\begin{pmatrix}1-J_{nr+n+1} & -stJ_{nr+n}\\ -J_{nr+n} & 1-stJ_{nr+n-1}\end{pmatrix}\\ &=\frac{1}{1-j_n+(-1)^n(st)^n}\begin{pmatrix}\left(1-stJ_{n-1}\right)\left(1-J_{nr+n+1}\right)-stJ_nJ_{nr+n} & -\left(1-stJ_{n-1}\right)stJ_{nr+n}+\left(1-stJ_{nr+n-1}\right)stJ_n\\ \left(1-J_{nr+n+1}\right)J_n-\left(1-J_{n+1}\right)J_{nr+n} & \left(1-J_{n+1}\right)\left(1-stJ_{nr+n-1}\right)-stJ_nJ_{nr+n}\end{pmatrix}.\end{aligned} \tag{24}$$

On the other hand, using (23) and (24), we get

$$\sum_{i=0}^{r}J_{ni}=\frac{\left(1-J_{nr+n+1}\right)J_n-\left(1-J_{n+1}\right)J_{nr+n}}{1-j_n+(-1)^n(st)^n}.$$

Corollary 14 For $n,r\geq 0$, the following results hold.

(i) $$\sum_{i=0}^{r}j_{ni}=\frac{\left(1-\frac{1}{2}j_n\right)\left(2-j_{nr+n}\right)-\frac{1}{2}(s+t)^2J_nJ_{nr+n}}{1-j_n+(-1)^n(st)^n},$$

(ii) $$\sum_{i=0}^{r}J_{ni}=\frac{\frac{1}{2}\left(2-j_{nr+n}\right)J_n-\left(1-\frac{1}{2}j_n\right)J_{nr+n}}{1-j_n+(-1)^n(st)^n}.$$

Proof. The proofs of (i) and (ii) are similar to Theorem 13.

Theorem 15 Let $n,r\geq 0$. It r is an even, then

$$\sum_{i=0}^{r}J_{ni}=\frac{\left(1+J_{n+1}\right)J_{nr+n}-\left(1+J_{nr+n+1}\right)J_n}{1+j_n+(-1)^n(st)^n}. \tag{25}$$

Proof. Let r be an even number. Then we have

$$I+\left(A^n\right)^{r+1}=\left(I+A^n\right)\sum_{i=0}^{r}(-1)^i\left(A^n\right)^i=\left(I+A^n\right)\sum_{i=0}^{r}(-1)^iA^{ni}.$$

By Lemma 8 (iii) and Lemma 9 (ii), we get

$$det\left(I+A^{n}\right)=1+stJ_{n-1}+J_{n+1}+stJ_{n-1}J_{n+1}-stJ_{n}^{2}=1+j_{n}+\left(-1\right)^{n}\left(st\right)^{n}.$$

Since $\det\left(I+A^{n}\right)\neq 0$ we can write

$$\left(I+A^{n}\right)^{-1}\left(I+\left(A^{n}\right)^{r+1}\right)=\sum_{i=0}^{r}A^{ni}=\begin{pmatrix}\sum_{i=0}^{r}J_{ni+1} & st\sum_{i=0}^{r}J_{ni}\\ \sum_{i=0}^{r}J_{ni} & st\sum_{i=0}^{r}J_{ni-1}\end{pmatrix}. \tag{26}$$

Since $\left(I+A^{n}\right)^{-1}=\dfrac{1}{1+j_{n}+\left(-1\right)^{n}\left(st\right)^{n}}\begin{pmatrix}1+stJ_{n-1} & -stJ_{n}\\ -J_{n} & 1+J_{n+1}\end{pmatrix}$ we have

$$\begin{aligned}&\left(I+A^{n}\right)^{-1}\left(I+A^{nr+n}\right)\\ &=\frac{1}{1+j_{n}+\left(-1\right)^{n}\left(st\right)^{n}}\begin{pmatrix}1+stJ_{n-1} & -stJ_{n}\\ -J_{n} & 1+J_{n+1}\end{pmatrix}\begin{pmatrix}1+J_{nr+n+1} & stJ_{nr+n}\\ J_{nr+n} & 1+stJ_{nr+n-1}\end{pmatrix}\\ &=\frac{1}{1+j_{n}+\left(-1\right)^{n}\left(st\right)^{n}}\begin{pmatrix}\left(1+stJ_{n-1}\right)\left(1+J_{nr+n+1}\right)-stJ_{n}J_{nr+n} & st\left(1+stJ_{n-1}\right)J_{nr+n}-st\left(1+stJ_{nr+n-1}\right)J_{n}\\ \left(1+J_{n+1}\right)J_{nr+n}-\left(1+J_{nr+n+1}\right)J_{n} & \left(1+J_{n+1}\right)\left(1+stJ_{nr+n-1}\right)-stJ_{n}J_{nr+n}\end{pmatrix}.\end{aligned} \tag{27}$$

On the other hand, using (26) and (27), we obtain

$$\sum_{i=0}^{r}J_{ni}=\frac{\left(1+J_{n+1}\right)J_{nr+n}-\left(1+J_{nr+n+1}\right)J_{n}}{1+j_{n}+\left(-1\right)^{n}\left(st\right)^{n}}.$$

Corollary 16 Let $n,r\geq 0$. It r is an even, then

(i) $\displaystyle\sum_{i=0}^{r}j_{ni}=\frac{\frac{1}{2}\left(2+j_{n}\right)\left(2+j_{nr+n}\right)-\frac{1}{2}\left(s+t\right)^{2}J_{n}J_{nr+n}}{1+j_{n}+\left(-1\right)^{n}\left(st\right)^{n}}$,

(ii) $\displaystyle\sum_{i=0}^{r}J_{ni}=\frac{\frac{1}{2}\left(2+j_{n}\right)J_{nr+n}-\frac{1}{2}\left(2+j_{nr+n}\right)J_{n}}{1+j_{n}+\left(-1\right)^{n}\left(st\right)^{n}}$.

Proof. The proofs of (i) and (ii) are similar to Theorem 15.

Theorem 17 For $m,k\geq 0$ and $n\geq 1$, the following results hold.

(i) $\displaystyle J_{mn+k}=\sum_{i=0}^{m}\binom{m}{i}\left(st\right)^{i}\left(J_{n-1}\right)^{i}\left(J_{n}\right)^{m-i}J_{m+k-i}$,

(ii) $\displaystyle j_{mn+k}=\sum_{i=0}^{m}\binom{m}{i}\left(st\right)^{i}\left(J_{n-1}\right)^{i}\left(J_{n}\right)^{m-i}j_{m+k-i}$.

Proof. Since Lemma 5 (i), we can write

$$\begin{aligned}\left(A^{n}\right)^{m}A^{k}&=\left(J_{n}A+stJ_{n-1}I\right)^{m}A^{k}\\ &=\sum_{i=0}^{m}\binom{m}{i}\left(J_{n}A\right)^{m-i}\left(stJ_{n-1}\right)^{i}A^{k}\end{aligned}$$

$$=\sum_{i=0}^{m}\binom{m}{i}(st)^{i}\left(J_{n-1}\right)^{i}\left(J_{n}\right)^{m-i}A^{m+k-i}. \tag{28}$$

By the power property of a matrix, we have

$$\left(A^{n}\right)^{m}A^{k}=A^{mn+k}. \tag{29}$$

By using (28) and (29), we obtain

$$J_{mn+k}=\sum_{i=0}^{m}\binom{m}{i}(st)^{i}\left(J_{n-1}\right)^{i}\left(J_{n}\right)^{m-i}J_{m+k-i}.$$

The proof of (ii) is similar to (i). Therefore, the proof is complete.

Discussion

In this paper, we show that X-matrix and Y-matrix satisfying to $X^{2}=(s-t)X+stI$ and $Y^{2}=(s+t)^{2}I$. Moreover, we establish particular cases of these matrices: A-matrix, B-matrix, and W-matrix that are useful to obtain many new identities of the modified (s,t) Jacobsthal and modified (s,t) Jacobsthal – Lucas numbers by using some properties of matrix operations.

Conclusions

In this paper, we consider the modified (s,t) Jacobsthal and modified (s,t) Jacobsthal – Lucas numbers, and we develop generating the 2x2 matrices A-matrix, B-matrix, and W-matrix. After that, we get some identities of the modified (s,t) Jacobsthal and modified (s,t) Jacobsthal – Lucas numbers, some identities of the relation between modified (s,t) Jacobsthal and modified (s,t) Jacobsthal – Lucas numbers, and some sum formulas for the modified (s,t) Jacobsthal and modified (s,t) Jacobsthal – Lucas numbers by using these matrices representation and some properties of matrix operations. Furthermore, we conjecture which this concept extends negative subscript and develops to the n x n matrix consisting of elements of other recurrence relations.

Acknowledgments

This research was supported by the Faculty of Science and Technology, Rajamangala University of Technology Thanyaburi (RMUTT), Pathum Thani, THAILAND.

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