วารสารวิทยาศาสตร์บูรพา ปีที่ 26 (ฉบับที่ 3) กันยายน – ธันวาคม พ.ศ. 2564
BURAPHA SCIENCE JOURNAL Volume 26 (No.3) September – December 2021 บทความวิจัย

ผลเฉลยคาบของระบบสมการเชิงผลต่างเชิงเส้นเป็นช่วงที่มีเงื่อนไขเริ่มต้นในแกน x ทางบวก

Periodic Solution of a Piecewise Linear System of Difference Equations with Initial Condition in Positive X - Axis

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Received : 23 November 2020
Revised : 20 February 2021
Accepted : 22 February 2021

บทคัดย่อ

ในบทความนี้ผู้วิจัยได้ศึกษาระบบสมการเชิงผลต่างเชิงเส้นเป็นช่วงที่มีเงื่อนไขเริ่มต้นเป็นแกน x ทางบวกซึ่งยังเป็นปัญหาปลายเปิด เราพบการมีอยู่จริงของ วง 5 และจุดสมดุล เราใช้การคำนวณด้วยการวนซ้ำและหลักอุปนัยเชิงคณิตศาสตร์ในการพิสูจน์ พฤติกรรมต่าง ๆ ของผลเฉลยของระบบสมการ เราทำการแบ่งแกน x ทางบวกออกเป็นช่วงย่อยและทำการค้นหาพฤติกรรมของผลเฉลยในแต่ละช่วงย่อย เราพบว่าสำหรับเงื่อนไขเริ่มต้นดังกล่าวตัวดึงดูดมีเพียง วง 5 หรือ จุดสมดุลเท่านั้น ยิ่งกว่านั้นเรายังพบขอบเขตเบซินของการดึงดูดของวง 5 และจุดสมดุล

คำสำคัญ : ระบบสมการเชิงเส้นเป็นช่วง ; ผลเฉลยคาบ ; จุดสมดุล ; สมการเชิงผลต่าง

Abstract

In this paper, we study a piecewise linear system of difference equations with initial condition in positive x-axis which remains an open problem. We find that there exist 5-cycles and equilibrium point. We use some direct iterative calculations and mathematical induction to prove the behaviors of solutions of the system. We separate positive x-axis into subintervals and investigate the behaviors of solutions in each of subintervals. We also find that for such initial condition the attractors are only 5-cycles and equilibrium point. Moreover, we reveal the boundary of basins of attractions for 5-cycles and equilibrium point.

Keywords : piecewise linear system ; periodic solution ; equilibrium point ; difference equation

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Introduction

The study of the dynamics of flows is strictly related to maps via the Poincaré sections and the related return maps. There are many phenomenon that seem discretely such as cell division, insect population, and Propagation of annual. Also two dimensional smooth maps describe many phenomena coming from population dynamics and economy theory. With above, the difference equations are more appropriate to model such situations. Lozi map (Lozi, 1978) is a well known piecewise linear map (PWL), one of the simplest cases of piecewise smooth map (PWS). It is a simplified version of Hénon map and has a strange attractor. There are many applications of PWS maps and PWL maps in models such as power electronic converters and switching circuits (see e.g. (Banerjee and Verghese, 2001; Zhusubaliyev &Mosekilde, 2003)), mechanical systems (Brogliato, 1999; Ma *et al.*, 2006; Ing. *et al.*, 2010). We know that multistability (Simpson, 2010; Zhusubaliyev *et al.*, 2008), and also infinitely many coexisting attractors (Simpson, 2014a,b) can occur in PWS map. Center bifurcations in a family of piecewise linear maps were cosidered in articles (Gardini & Tikjha, 2019; Tikjha & Gardini, 2020) and also a transition case between invertibility and non-invertibility of piecewise linear map were found in article (Gardini & Tikjha, 2020). An open problem about a system were mentioned in (Grove *et al.*,2012):

$$\begin{cases} x_{n+1} = |x_n| + ay_n + b \\ y_{n+1} = x_n + c|y_n| + d \end{cases}, n = 0,1,2,\ldots \tag{1}$$

where parameters a,b,c and d are in $\{-1,0,1\}$ and the initial condition $(x_0, y_0) \in R^2$. This family of systems was inspired by the Devaney's Gingerbreadman Map and the Lozi Map (Lozi, 1978). For other systems of this form see articles (Devaney,1984; Botella-Soler *et al.*, 2011). Several articles investigate the open problem for example: Gove et al. (2012) found that every solution of a special case of system (1) is eventually prime period-3 solutions except for the unique equilibrium solution. In article (Tikjha *et al.*, 2010; 2015; 2017) and (Tikjha & Lapiere, 2020), they studied some special cases of system (1), showed that there are periodic attractors. They showed that every solution is eventually either that attractors or equilibrium point by using direct calculation and inductive statement. Our ultimate goal to make generalized on parameter b of special case of system (1) which is as following,

$$\begin{cases} x_{n+1} = |x_n| - y_n + b \\ y_{n+1} = x_n + |y_n| + 1 \end{cases}, n = 0,1,2,\ldots \tag{2}$$

where b is any real number. Krinket (2015) studied a special case of system (2) by taking $b = -1$, she found that solution is eventually prime period 4 with specific initial condition on $y-$ axis. Tikjha & Piasu (2020) studied the solution of special case of system (2) by taking $b=-3$, with initial condition being a specific region in first quadrant is eventually equilibrium point or prime period 4. Jittbrurus & Tikjha (2020) studied the solution of special case of system (2) by taking $b=-7$,

$$\begin{cases} x_{n+1} = |x_n| - y_n - 7 \\ y_{n+1} = x_n - |y_n| + 1 \end{cases}, n = 0,1,2,\ldots \tag{3}$$

with initial condition on negative y-axis. To predict the global behavior of system (2), we have to gather as much results as of special cases to system (2). So we continue to investigate system (3) with initial condition on positive x-axis.

Methods

The following definitions (Grove &Ladas, 2005) are used in this article. A solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of a system of difference equations is called *eventually periodic with prime period p* or *eventually prime period p solution* if there exists an integer $N > 0$ and p is the smallest positive integer such that $\{(x_n, y_n)\}_{n=0}^{\infty}$ is periodic with period p; that is, $(x_{n+p}, y_{n+p}) = (x_n, y_n)$ for all $n \geq N$. The 5 consecutive points of the solution is called a *5-cycle* of system (3). We denote $\{((a,b),(c,d),(e,f),(g,h),(i,j))\}$ as 5-cycle which consists of 5 consecutive points: $(a,b),(c,d),(e,f),(g,h)$ and (i,j) in xy plane. It is worth noting that solution is eventually periodic with period p when *orbit* (forward iterations) contains a member of the cycle. We will show that every solution of the system (3) with positive x-axis initial condition is eventually either equilibrium point or 5-cycle by finding pattern of solutions and then verify that the closed form is true by using mathematical induction.

Results

In this section, we will investigate the global behavior of system (3) with initial condition in positive x-axis. The system (3) has a unique equilibrium point which is $(-1,-5)$ from solving the system of equations :

$$\begin{cases} \bar{x} = |\bar{x}| - \bar{y} - 7 \\ \bar{y} = \bar{x} - |\bar{y}| + 1 \end{cases}.$$

The system also have periodic solutions as 5-cycles that are $P_{5.1}$ and $P_{5.2}$ as follows:

$$P_{5.1}=\{((-5,-7),(5,-11),(9,-5),(7,5),(-5,3))\}$$ and
$$P_{5.2}=\{((15/7,-57/7),(23/7,-5),(9/7,-5/7),(-5,11/7),(-25/7,-39/7))\}.$$

We will show that the attractors, with specific initial condition in positive x- axis, of this system are only the equilibrium and 5-cycles and we could reveal the border of basin of attraction between the attractors. The proof of theorem is looking for behaviors of solutions of the system by separating positive x- axis into four main intervals which are $(0,1/4],(1/4,3/4),[3/4,7)$ and $[7,\infty)$.

The most easiest intervals that we will investigate is when initial condition (x_0,y_0) with $x_0\in[7,\infty)$ and $y_0=0$. Then for the initial condition $x_0\in[7,\infty)$ and $y_0=0$, we have the first iteration:

$$\begin{cases} x_1=|x_0|-y_0-7=x_0-7\geq 0 \\ y_1=x_0-|y_0|+1=x_0+1>0 \end{cases}.$$

Then we have the second iteration $(x_2,y_2)=(-15,-7)$ and the sixth iteration $(x_6,y_6)=(-5,3)\in P_{5.1}$. So the solution is prime period 5 within 6 iterations as the following lemma.

Lemma 1 Let $\{(x_n,y_n)\}_{n=1}^{\infty}$ be solutions of system (3) and initial condition (x_0,y_0) is in $\{(x,y)\mid x\geq 7, y=0\}$. Then the solution is 5-cycle $P_{5.1}$ within 6 iterations.

The second interval that we will investigate is when initial condition (x_0,y_0) with $x_0\in[3/4,7)$ and $y_0=0$. Then for the initial condition $x_0\in[3/4,7)$ and $y_0=0$, we have the list of the closed form of solutions as follows:

$$\begin{cases} x_1=x_0-7<0 \\ y_1=x_0+1>0 \end{cases},\begin{cases} x_2=-2x_0-1<0 \\ y_2=-7 \end{cases},\begin{cases} x_3=2x_0+1>0 \\ y_3=-2x_0-7<0 \end{cases},\begin{cases} x_4=4x_0+1>0 \\ y_4=-5 \end{cases},\begin{cases} x_5=4x_0-1>0 \\ y_5=4x_0-3\geq 0 \end{cases},$$ and

then $(x_6,y_6)=(-5,3)\in P_{5.1}$.

So the solution is prime period 5 within 6 iterations as the following lemma.

Lemma 2 Let $\{(x_n, y_n)\}_{n=1}^{\infty}$ be solutions of system (3) and initial condition (x_0, y_0) is in $\{(x, y) \mid 3/4 \le x < 7, y = 0\}$. Then the solution is 5-cycle $P_{5.1}$ within 6 iterations.

The third interval that we will investigate is when initial condition (x_0, y_0) with $x_0 \in (0, 1/4]$ and $y_0 = 0$. Then for the initial condition $x_0 \in (0, 1/4]$ and $y_0 = 0$, we have the list of the closed form which the first four iterations are the same as the previous lemma and the closed form of the remaining solutions are as follows:

$\begin{cases} x_5 = 4x_0 - 1 \le 0 \\ y_5 = 4x_0 - 3 < 0 \end{cases}$, $\begin{cases} x_6 = -8x_0 - 3 < 0 \\ y_6 = 8x_0 - 3 < 0 \end{cases}$ and then $(x_7, y_7) = (-1, -5)$. So the solution is equilibrium point within 7 iterations as the following lemma.

Lemma 3 Let $\{(x_n, y_n)\}_{n=1}^{\infty}$ be solutions of system (3) and initial condition (x_0, y_0) is in $\{(x, y) \mid 0 < x \le 1/4, y = 0\}$. Then the solution is equilibrium point within 7 iterations.

The last interval that we will investigate is when initial condition (x_0, y_0) with $x_0 \in (1/4, 3/4)$ and $y_0 = 0$. The closed form of solutions is shown in the proof of the following lemma.

Lemma 4 Let $\{(x_n, y_n)\}_{n=1}^{\infty}$ be solutions of system (3) and initial condition (x_0, y_0) is in $L = \{(x, y) \mid 1/4 < x < 3/4, y = 0\}$. Then the solution is eventually either 5-cycles or equilibrium point $(-1, -5)$.

Proof. We have a closed form of the first four iterations which are the same as those in Lemma 3 and $\begin{cases} x_5 = 4x_0 - 1 > 0 \\ y_5 = 4x_0 - 3 < 0 \end{cases}$. The closed form of remaining solutions using the following sequences:

$$l_n = \frac{4 \times 2^{3n-1} - 9}{7 \times 2^{3n-1}},\ u_n = \frac{4 \times 2^{3n-1} + 5}{7 \times 2^{3n-1}},\ a_n = \frac{4 \times 2^{3n} - 11}{7 \times 2^{3n}},\ b_n = \frac{4 \times 2^{3n+1} - 15}{7 \times 2^{3n+1}},\ c_n = \frac{4 \times 2^{3n+2} - 23}{7 \times 2^{3n+2}}$$

$\delta_n = \dfrac{4 \times 2^{3n} - 11}{7}$. Let $P(n)$ be the following statement:

"for $x_0 \in (l_n, u_n)$,

$x_{5n+1} = -5$, $y_{5n+1} = 2^{3n} x_0 - \delta_n$

if $x_0 \in (l_n, a_n]$ then $y_{5n+1} \le 0$

$x_{5n+2} = -2^{3n} x_0 + \delta_n - 2 < 0$, $y_{5n+2} = 2^{3n} x_0 - \delta_n - 4 < 0$

$x_{5n+3} = -1$, $y_{5n+3} = -5$

if $x_0 \in (a_n, u_n)$ then $y_{5n+1} > 0$ and so

$x_{5n+2} = -2^{3n} x_0 + \delta_n - 2 < 0$, $y_{5n+2} = -2^{3n} x_0 + \delta_n - 4 < 0$

$x_{5n+3} = 2^{3n+1} x_0 - 2\delta_n - 1$, $y_{5n+3} = -2^{3n+1} x_0 + 2\delta_n - 5 < 0$

if $x_0 \in (a_n, b_n]$ then $x_{5n+3} \le 0$ and so

$x_{5n+4} = -1$, $y_{5n+4} = -5$

if $x_0 \in (b_n, u_n)$ then $x_{5n+3} > 0$ and so

$x_{5n+4} = 2^{3n+2} x_0 - 4\delta_n - 3$, $y_{5n+4} = -5$

if $x_0 \in (b_n, c_n]$ then $x_{5n+4} \le 0$ and so

$x_{5n+5} = -2^{3n+2} x_0 + 4\delta_n + 1 < 0$, $y_{5n+5} = 2^{3n+2} x_0 - 4\delta_n - 7 < 0$

$x_{5n+6} = -1$, $y_{5n+6} = -5$

if $x_0 \in (c_n, u_n)$ then $x_{5n+4} > 0$ and so

$x_{5n+5} = 2^{3n+2} x_0 - 4\delta_n - 5$, $y_{5n+5} = 2^{3n+2} x_0 - 4\delta_n - 7$

if $x_0 \in (c_n, l_{n+1}]$ then $x_{5n+5} \le 0$ and $y_{5n+5} < 0$ and so

$x_{5n+6} = -2^{3n+3} x_0 + 8\delta_n + 5 < 0$, $y_{5n+6} = 2^{3n+3} x_0 - 8\delta_n - 11 < 0$

$x_{5n+7} = -1$, $y_{5n+7} = -5$

if $x_0 \in [u_{n+1}, u_n)$ then $x_{5n+5} > 0$ and $y_{5n+5} \ge 0$ and so

$x_{5n+6} = -5$, $y_{5n+6} = 3$

if $x_0 \in (l_{n+1}, u_{n+1})$ then $x_{5n+5} > 0$ and $y_{5n+5} < 0$".

Firstly, we shall show that $P(1)$ is true. By letting $x_0 \in (l_1, u_1) = (1/4, 3/4)$ and we know that $x_5 = 4x_0 - 1 > 0$ and $y_5 = 4x_0 - 3 < 0$. Then $x_6 = -5, y_6 = 8x_0 - 3 = 2^{3(1)} x_0 - \delta_1$.

If $x_0 \in (l_1, a_1] = (1/4, 3/8]$ then $y_6 = 8x_0 - 3 \le 0$. Thus $x_7 = -8x_0 + 1 = -2^{3(1)} x_0 + \delta_1 - 2 < 0$ and $y_7 = 8x_0 - 7 = 2^{3(1)} x_0 - \delta_1 - 4 < 0$. So $(x_8, y_8) = (-1, -5)$.

If $x_0 \in (a_1, u_1) = (3/8, 3/4)$ then $y_6 = 8x_0 - 3 > 0$. Thus
$x_7 = -8x_0 + 1 = -2^{3(1)} x_0 + \delta_1 - 2 < 0$ and $y_7 = -8x_0 - 1 = -2^{3(1)} x_0 + \delta_1 - 4 < 0$. So
$x_8 = 16x_0 - 7 = 2^{3(1)+1} x_0 - 2\delta_1 - 1$ and $y_8 = -16x_0 + 1 = -2^{3(1)+1} x_0 + 2\delta_1 - 5 < 0$.

If $x_0 \in (a_1, b_1] = (3/8, 7/16]$ then $x_8 = 16x_0 - 7 \le 0$. So $(x_9, y_9) = (-1, -5)$.
If $x_0 \in (b_1, u_1) = (7/16, 3/4)$ then $x_8 = 16x_0 - 7 > 0$. So $x_9 = 32x_0 - 15 = 2^{3(1)+2} x_0 - 4\delta_1 - 3$ and $y_9 = -5$.

If $x_0 \in (b_1, c_1] = (7/16, 15/32]$ then $x_9 = 32x_0 - 15 \le 0$. So
$x_{10} = -32x_0 + 13 = -2^{3(1)+2} x_0 + 4\delta_1 + 1 < 0$ and $y_{10} = 32x_0 - 19 = 2^{3(1)+2} x_0 - 4\delta_1 - 7 < 0$. So $(x_{11}, y_{11}) = (-1, -5)$.

If $x_0 \in (c_1, u_1) = (15/32, 3/4)$ then $x_9 = 32x_0 - 15 > 0$. So
$x_{10} = 32x_0 - 17 = 2^{3(1)+2} x_0 - 4\delta_1 - 5$ and $y_{10} = 32x_0 - 19 = 2^{3(1)+2} x_0 - 4\delta_1 - 7$.

If $x_0 \in (c_1, l_2] = (15/32, 17/32]$ then $x_{10} = 32x_0 - 17 \le 0$ and $y_{10} = 32x_0 - 19 < 0$. So
$x_{11} = -64x_0 + 29 = -2^{3(1)+3} x_0 + 8\delta_1 + 5 < 0$ and $y_{11} = 64x_0 - 35 = 2^{3(1)+3} x_0 - 8\delta_1 - 11 < 0$. Then $(x_9, y_9) = (-1, -5)$.

If $x_0 \in [u_2, u_1) = [19/32, 3/4)$ then $x_{10} = 32x_0 - 17 > 0$ and $y_{10} = 32x_0 - 19 \ge 0$. So $(x_{11}, y_{11}) = (-5, 3)$.

If $x_0 \in (l_2, u_2) = (17/32, 19/32)$ then $x_{10} = 32x_0 - 17 > 0$ and $y_{10} = 32x_0 - 19 < 0$.
Hence $P(1)$ is true. Now we suppose further that $P(k)$ is true. We have $x_{5k+5} = 2^{3k+2} x_0 - 4\delta_k - 5 > 0$ and $y_{5k+5} = 2^{3k+2} x_0 - 4\delta_k - 7 < 0$ for $x_0 \in (l_{k+1}, u_{k+1}) = \left(\frac{4 \times 2^{3k+2} - 9}{7 \times 2^{3k+2}}, \frac{4 \times 2^{3k+2} + 5}{7 \times 2^{3k+2}} \right)$. Then
$x_{5(k+1)+1} = x_{5k+6} = -5$ and $y_{5(k+1)+1} = y_{5k+6} = 2^{3k+3} x_0 - (8\delta_k + 11) = 2^{3k+3} x_0 - \delta_{k+1}$.

We note that $8\delta_k + 11 = 8\left(\frac{4 \times 2^{3k} - 11}{7} \right) + 11 = \frac{32 \times 2^{3k} - 88 + 77}{7} = \frac{4 \times 2^{3k+3} - 11}{7} = \delta_{k+1}$.

If $x_0 \in (l_{k+1}, a_{k+1}] = \left(\frac{4\times 2^{3k+2}-9}{7\times 2^{3k+2}}, \frac{4\times 2^{3k+3}-11}{7\times 2^{3k+3}}\right]$ then $y_{5k+6} = 2^{3k+3}x_0 - \left(\frac{4\times 2^{3k+3}-11}{7}\right) \le 0$. We determine the sign of y_{5k+6} by substituting l_{k+1} and a_{k+1} into x_0 of linear function y_{5k+6} as following:

$y_{5k+6}(l_{k+1}) = 2^{3k+3}\left(\frac{4\times 2^{3k+2}-9}{7\times 2^{3k+2}}\right) - \left(\frac{4\times 2^{3k+3}-11}{7}\right) = \left(\frac{2^{3k+5}-18}{7}\right) - \left(\frac{2^{3k+5}-11}{7}\right) = -1$ and

$y_{5k+6}(a_{k+1}) = 2^{3k+3}\left(\frac{4\times 2^{3k+3}-11}{7\times 2^{3k+3}}\right) - \left(\frac{4\times 2^{3k+3}-11}{7}\right) = 0$. From now on, we will determine the sign of solutions by using this method. Then $x_{5k+7} = -2^{3k+3}x_0 + \delta_{k+1} - 2 < 0$ and $y_{5k+7} = 2^{3k+3}x_0 - \delta_{k+1} - 4 < 0$ and so $(x_{5k+8}, y_{5k+8}) = (-1,-5)$ as require.

If $x_0 \in (a_{k+1}, u_{k+1}) = \left(\frac{4\times 2^{3k+3}-11}{7\times 2^{3k+3}}, \frac{4\times 2^{3k+2}+5}{7\times 2^{3k+2}}\right)$ then $y_{5(k+1)+1} = y_{5k+6} > 0$. Then

$x_{5k+7} = -2^{3k+3}x_0 + \delta_{k+1} - 2 < 0$ and $y_{5k+7} = -2^{3k+3}x_0 + \delta_{k+1} - 4 < 0$ and so

$x_{5k+8} = 2^{3k+4}x_0 - 2\delta_{k+1} - 1$ and $y_{5k+8} = -2^{3k+4}x_0 + 2\delta_{k+1} - 5 < 0$.

If $x_0 \in (a_{k+1}, b_{k+1}] = \left(\frac{4\times 2^{3k+3}-11}{7\times 2^{3k+3}}, \frac{4\times 2^{3k+4}-15}{7\times 2^{3k+4}}\right]$ then $x_{5(k+1)+3} = x_{5k+8} \le 0$, So $(x_{5k+9}, y_{5k+9}) = (-1,-5)$ as require.

If $x_0 \in (b_{k+1}, u_{k+1}) = \left(\frac{4\times 2^{3k+4}-15}{7\times 2^{3k+4}}, \frac{4\times 2^{3k+2}+5}{7\times 2^{3k+2}}\right)$ then $x_{5(k+1)+3} = x_{5k+8} > 0$. So

$x_{5k+9} = 2^{3k+5}x_0 - 4\delta_{k+1} - 3$ and $y_{5k+9} = -5$.

If $x_0 \in (b_{k+1}, c_{k+1}] = \left(\frac{4\times 2^{3k+4}-15}{7\times 2^{3k+4}}, \frac{4\times 2^{3k+5}-23}{7\times 2^{3k+5}}\right]$ then $x_{5(k+1)+4} = x_{5k+9} \le 0$. Thus

$x_{5k+10} = -2^{3k+5}x_0 + 4\delta_{k+1} + 1 < 0$ and $y_{5k+10} = 2^{3k+5}x_0 - 4\delta_{k+1} - 7 < 0$ and so $(x_{5k+11}, y_{5k+11}) = (-1,-5)$ as require.

If $x_0 \in (c_{k+1}, u_{k+1}) = \left(\frac{4\times 2^{3k+5}-23}{7\times 2^{3k+5}}, \frac{4\times 2^{3k+2}+5}{7\times 2^{3k+2}}\right)$ then $x_{5(k+1)+4} = x_{5k+9} > 0$. Thus

$x_{5k+10} = 2^{3k+5}x_0 - 4\delta_{k+1} - 5$ and $y_{5k+10} = 2^{3k+5}x_0 - 4\delta_{k+1} - 7$.

If $x_0 \in (c_{k+1}, l_{k+2}] = \left(\dfrac{4\times 2^{3k+5}-23}{7\times 2^{3k+5}}, \dfrac{4\times 2^{3k+5}-9}{7\times 2^{3k+5}}\right]$ then $x_{5(k+1)+5} = x_{5k+10} \le 0$ and $y_{5(k+1)+5} = y_{5k+10} < 0$.
Thus $x_{5k+11} = -2^{3k+6}x_0 + 8\delta_{k+1} + 5 < 0$ and $y_{5k+11} = 2^{3k+6}x_0 - 8\delta_{k+1} - 11 < 0$ and so $(x_{5k+12}, y_{5k+12}) = (-1,-5)$ as require.
If $x_0 \in [u_{k+2}, u_{k+1}) = \left[\dfrac{4\times 2^{3k+5}+5}{7\times 2^{3k+5}}, \dfrac{4\times 2^{3k+2}+5}{7\times 2^{3k+2}}\right)$ then $x_{5(k+1)+5} = x_{5k+10} > 0$ and $y_{5(k+1)+5} = y_{5k+10} \ge 0$.
Thus $(x_{5k+11}, y_{5k+11}) = (-5,3)$ as require.

If $x_0 \in (l_{k+2}, u_{k+2}) = \left(\dfrac{4\times 2^{3k+5}-9}{7\times 2^{3k+5}}, \dfrac{4\times 2^{3k+5}+5}{7\times 2^{3k+5}}\right)$ then $x_{5(k+1)+5} = x_{5k+10} > 0$ and $y_{5(k+1)+5} = y_{5k+10} < 0$.
Hence $P(k+1)$ is true. By induction we conclude that $P(n)$ is true for every positive integer n. By the inductive statement $P(n)$, we can further conclude the following statement.

1. The solution is eventually equilibrium point when $x_0 \in (l_n, a_n] \cup (a_n, b_n] \cup (b_n, c_n] \cup (c_n, l_{n+1}]$ and $y_0 = 0$ for some n.
2. The solution is eventually 5-cycle $P_{5.1}$ when $x_0 \in [u_{n+1}, u_n)$ and $y_0 = 0$ for some n.

We note that sequence a_n, b_n, c_n, l_n are in the left side of their limits and sequences u_n is in the right side of its limit. We also note that the sequences a_n, b_n, c_n, u_n, l_n have the same limit:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = \lim_{n\to\infty} l_n = \lim_{n\to\infty} u_n = \frac{4}{7}.$$

If we exactly choose the initial condition at $(x_0, y_0) = (4/7, 0)$, then the third iteration is $(x_3, y_3) = (15/7, -57/7) \in P_{5.2}$. Therefore we can conclude that every solution of system (3) with initial in L is eventually equilibrium point or 5-cycles. □

By above lemmas, we can conclude all behaviors of solutions of the system (3) as the following theorem.

Theorem 5 Let $\{(x_n, y_n)\}_{n=1}^{\infty}$ be solutions of system (3) and initial condition (x_0, y_0) is in positive x- axis. Then the solution is eventually either 5-cycle or equilibrium point (-1,-5).

Discussion

The behaviors of solutions of system (3) in this study is agree to results of Jittbrurus and Tikjha (2020) however the closed form of solutions are difference because we take different initial condition. By the results of Jittbrurus and Tikjha (2020) and our results, we can conclude that the solution with initial condition in positive x-axis and negative y- axis is eventually equilibrium point or 5-cycles. We believe that our results will be a tool for studying this system with another initial condition region of R^2. It is possible to have only equilibrium point and 5-cycles as attractors. We do still not confirm that the attractors are only equilibrium point and 5-cyles until knowing all behaviors with initial condition of every point in R^2.

Conclusion

By separating initial condition along the positive x-axis into subintervals and looking behaviors of solutions of system (3) with initial condition in each interval, we can predict the behaviors of solutions of system (3). When $x_0 \in [3/4, \infty)$ and $y_0 = 0$ solutions will be 5-cycle $P_{5.1}$ within 6 iterations, when $x_0 \in (0, 1/4]$ and $y_0 = 0$ solutions will be equilibrium point $(-1, -5)$ within 7 iterations and when $x_0 \in (1/4, 3/4)$ and $y_0 = 0$ solutions will be 5-cycle or equilibrium point depending on choosing the initial condition. The boundary of basins of attraction of 5-cycles and equilibrium point is a point $(4/7, 0)$. Moreover, for initial point $(4/7, 0)$, the solution is eventually 5-cycle $P_{5.2}$. The proof of Lemma 4, $x_0 \in (1/4, 3/4)$ and $y_0 = 0$, requires an inductive statement to be a tool to verify that the result is true.

Acknowledgements

The authors are supported by the Centre of Excellence in Mathematics, National Research Council of Thailand and Pibulsongkram Rajabhat University.

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