วารสารวิทยาศาสตร์บูรพา ปีที่ 26 (ฉบับที่ 3) กันยายน – ธันวาคม พ.ศ. 2564
BURAPHA SCIENCE JOURNAL Volume 26 (No.3) September – December 2021 บทความวิจัย

การกำกับมหัศจรรย์แบบ S ของยูเนียนของกราฟหลายส่วนบริบูรณ์ที่ไม่มีจุดร่วมกัน

S – magic Labelings of Union of Disjoint Complete Multipartite Graphs

ธนวัฒน์ วิเชียรไพศาล[1] และ อุทุมพร มาโต[2*]

Tanawat Wichianpaisarn[1] and Uthoomporn Mato[2*]

[1]ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ประยุกต์ มหาวิทยาลัยเทคโนโลยีพระจอมเกล้าพระนครเหนือ

[2]ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยศรีนครินทรวิโรฒ

[1]Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok

[2]Department of Mathematics, Faculty of Science, Srinakharinwirot University

Received : 4 November 2020

Revised : 23 February 2021

Accepted : 1 March 2021

บทคัดย่อ

การกำกับมหัศจรรย์แบบ S ของกราฟ G คือฟังก์ชันหนึ่งต่อหนึ่ง f จากเซตของจุด $V(G)$ ไปยังเซตของจำนวนเต็มบวก S โดยที่ $|S| = |V(G)|$ ซึ่ง $\sum_{v \in N_G(u)} f(v) = k$ สำหรับจุด $u \in V(G)$ โดยที่ k เป็นค่าคงตัวและ $N_G(u)$ แทนเซตของจุดทั้งหมดใน G ที่ประชิดกับจุด u จะกล่าวว่ากราฟ G เป็นมหัศจรรย์แบบ S ถ้ากราฟ G มีการกำกับมหัศจรรย์แบบ S ในงานวิจัยนี้ เราได้แสดงเงื่อนไขที่จำเป็นและเพียงพอของการมีการกำกับมหัศจรรย์แบบ S ของยูเนียนของกราฟหลายส่วนบริบูรณ์ที่ไม่มีจุดร่วมกัน G และได้แสดงด้วยว่า กราฟ G ที่เพิ่มหนึ่งเส้น e จะไม่เป็นมหัศจรรย์แบบ S ถ้าจุดปลายทั้งสองของ e มาจากเซตแบ่งส่วนเดียวกัน

คำสำคัญ : มหัศจรรย์แบบ S ; การกำกับของกราฟ ; กราฟหลายส่วน

Abstract

An S – magic labeling of a graph G is a one to one map f from the vertex set $V(G)$ to a set of positive integers S with $|S| = |V(G)|$, such that $\sum_{v \in N_G(u)} f(v) = k$ for any $u \in V(G)$ where k is a constant and $N_G(u)$ is the set of vertices in G adjacent to u. A graph G is S – magic if G admits an S – magic labeling. In this paper, we show a necessary and sufficient condition for the existence of an S – magic labelings of a union of disjoint complete multipartite graphs G, we also prove that the graph G adding one edge e is not S – magic if two endpoints of e are from the same partite set.

Keywords : *S*-magic ; graph labeling ; multipartite graph

*Corresponding author. E-mail : uthoomporn@g.swu.ac.th

Introduction

Throughout this paper we consider finite, simple and undirected graphs. Notations and terminologies not defined here are followed from West (West, 2001).

Let $G=(V(G),E(G))$ be a graph. The *neighborhood* of a vertex u in a graph G is the set of vertices adjacent to u, and is denoted by $N_G(u)$. Let S be a set of positive integers where $|S|=|V(G)|$. An $S-$ *magic labeling* of G is a bijection $f:V(G)\rightarrow S$ such that $\sum_{v\in N_G(u)} f(v)=k$ for all $u\in V(G)$, where k is a constant. The constant k is called an $S-$ *magic constant*. If G admits an *S*-magic labeling, then the graph G is an $S-$ *magic graph* (or G is *S*-magic). For convenience, we denote $sum(f[V])=\sum_{v\in V} f(v)$ where V is a set of vertices.

If we restrict the labeling set of f to be $\{1,2,\cdots,n\}$ where $n=|V(G)|$ under the same condition of $S-$ magic labelings, then we say that f is a $\sum-$ labeling. A $\sum-$ graph is a graph which admits a $\sum-$ labeling.

In 1994, Vilfred (Vilfred, 1994) introduced the $\sum-$ labeling of graphs. The concept of $\sum-$ labeling was studied by many authors and several terminologies. For examples, Miller et al. (Miller *et al*., 2003) used the term $1-$ vertex magic labeling, Sugeng et al. (Sugeng *et al*., 2009) and Sankar et al. (Sankar *et al*., 2016) used the term distance magic labeling. Many results on distance magic labelings has been collected by Arumugam et al. (Arumugam *et al*., 2011).

In 2015, Godinho and Singh (Godinho and Singh, 2015) defined an $S-$ magic labeling which is the generalization of the idea of $\sum-$ labelings. They obtain many families of $S-$ magic graphs such as a complete graph K_n, a cycle C_n and a complete $r-$ partite graph $K_{n_1,n_2,\ldots,n_r}$. They also studied some properties of this labeling.

Many types of labeling have been studied and a survey of graph labelings can be found in (Gallian, 2017).

The main focus of this paper is to find some necessary and sufficient conditions for the existence of an $S-$ magic labelings of a union of disjoint complete multipartite graphs G and show that the graph G adding one edge is not $S-$ magic.

Methods

First, we studied the definitions of S – magic labelings of graphs and some properties that concern. Then we focused on S – magic labelings of the specific graphs such as the complete graphs which had been shown in the Godinho and Singh's paper . We collected the graphs that admit an S – magic labelings and found a new graph that no one studied in term of this labelings. Then we used some techniques of Godinho and Singh's paper as the basis on construction of S – magic labelings of the union of disjoint complete multipartite graphs.

Results

Let $m \geq 2$ be an integer. The union of disjoint m copies of G is denoted by mG. For $t \in \{1, 2, \cdots, m\}$, let $V_1^t, V_2^t, \cdots, V_r^t$ be the partite sets in the t^{th} copy of $mK_{n_1, n_2, \ldots, n_r}$ where $V_k^t = \{v_{k,1}^t, v_{k,2}^t, \cdots, v_{k,n_k}^t\}$ for all $k \in \{1, 2, \cdots, r\}$. That is $V\left(mK_{n_1, n_2, \ldots, n_r}\right) = \bigcup_{t=1}^{m} \bigcup_{k=1}^{r} V_k^t$.

Lemma 2.1. Let $f : V(mK_{n_1, n_2, \ldots, n_r}) \to S$ be a bijection where $S \subset$ □. The graph $G = mK_{n_1, n_2, \ldots, n_r}$ is S – magic with labeling f if and only if the sum of the labels of all the vertices in any two partite sets of G are equal.

Proof. Let f be an S-magic labeling of G. For $t \in \{1, 2, \cdots, m\}$, let $x \in V_i^t$ and $y \in V_j^t$ for some $i, j \in \{1, 2, \cdots, r\}$. Then $sum(f[N_G(x)]) = sum(f[N_G(y)])$

$$sum(f[\bigcup_{l=1}^{r} V_l^t]) - sum(f[V_i^t]) = sum(f[\bigcup_{l=1}^{r} V_l^t]) - sum(f[V_j^t])$$

$$sum(f[V_i^t]) = sum(f[V_j^t]).$$

For $t \in \{1, 2, \cdots, m\}$, let $sum(f[V_i^t]) = k_t$ for all $i \in \{1, 2, \cdots, r\}$.

Let $w \in V_i^p$ and $z \in V_j^q$ for some $i, j \in \{1, 2, \cdots, r\}$ and $p, q \in \{1, 2, \cdots, m\}$ where $p \neq q$. Then

$$sum(f[N_G(w)]) = sum(f[N_G(z)])$$

$$k_p(r-1) = k_q(r-1)$$

$$k_p = k_q.$$

Hence $sum(f[V_i^p]) = sum(f[V_j^q])$ for all $i, j \in \{1, 2, \cdots, r\}$ and $p, q \in \{1, 2, \cdots, m\}$.

Conversely, let $f : V(G) \to S$ be a bijection such that $sum(f[V_i^t]) = k$ for all $i \in \{1, 2, \cdots, r\}$ and $t \in \{1, 2, \cdots, m\}$. Thus $sum(f[N_G(u)]) = k(r-1)$ for all $u \in V(G)$. Then f is an S – magic labeling of G, and hence G is S – magic.

■

In 2015, Godinho and Singh (Godinho and Singh, 2015) found the necessary and sufficient conditions for the existence of $S-$ magic labelings of complete multipartite graphs as the following theorem.

Theorem 2.2. (Godinho and Singh, 2015) The complete r partite graph $K_{n_1, n_2, \ldots, n_r}$ where $n_1 \le n_2 \le \ldots \le n_r$ is $S-$ magic if and only if $n_2 \ge 2$.

We extend the graph in Theorem 2.2 to be a graph having many components. Observe that if G is $S-$ magic, then the graph mG may not be $S-$ magic, so we try to find some conditions for the existence of an $S-$ magic labeling of the union of disjoint complete multipartite graphs. The result as the following.

Theorem 2.3. The graph $G = mK_{n_1, n_2, \ldots, n_r}$ where $n_1 \le n_2 \le \ldots \le n_r$ and $m \ge 2$ is $S-$ magic if and only if $n_1 \ge 2$.

Proof. Let $n_1 = 1$. Suppose that G is $S-$ magic with labeling f. By Lemma 2.1, the sum of the labels of all the vertices in any two partite sets of G are equal. Since $V_1^1 = \{v_{1,1}^1\}$ and $V_1^2 = \{v_{1,1}^2\}$, we have $f(v_{1,1}^1) = f(v_{1,1}^2)$, a contradiction. Thus G is not $S-$ magic.

Conversely, assume that $n_1 \ge 2$. We define $h : V(G) \to \mathbb{N}$ as follows :

For $k \in \{1, 2, \cdots, r\}$, $p \in \{1, 2, \cdots, m\}$ and $i \in \{1, 2, \cdots, n_k - 1\}$, let

$$h(v_{k,i}^p) = \begin{cases} i + (p-1)(n_k - 1) & \text{if } k = 1, \\ i + (p-1)(n_k - 1) + m\sum_{j=1}^{k-1}(n_j - 1) & \text{if } k \in \{2, 3, \cdots, r\}. \end{cases}$$

Note that the sequence

$$\begin{array}{l} h(v_{1,1}^1),\ h(v_{1,2}^1),\ \cdots,\ h(v_{1,n_1-1}^1), \\ h(v_{1,1}^2),\ h(v_{1,2}^2),\ \cdots,\ h(v_{1,n_1-1}^2), \\ \qquad\qquad\vdots \\ h(v_{1,1}^m),\ h(v_{1,2}^m),\ \cdots,\ h(v_{1,n_1-1}^m), \end{array}$$

$$\begin{array}{l} h(v_{2,1}^1),\ h(v_{2,2}^1),\ \cdots,\ h(v_{2,n_2-1}^1), \\ h(v_{2,1}^2),\ h(v_{2,2}^2),\ \cdots,\ h(v_{2,n_2-1}^2), \\ \qquad\qquad\vdots \\ h(v_{2,1}^m),\ h(v_{2,2}^m),\ \cdots,\ h(v_{2,n_2-1}^m), \end{array}$$

$$\vdots$$
$$h(v^1_{r,1}),\ h(v^1_{r,2}),\cdots,\ h(v^1_{r,n_r-1}),$$
$$h(v^2_{r,1}),\ h(v^2_{r,2}),\cdots,\ h(v^2_{r,n_r-1}),$$
$$\vdots$$
$$h(v^m_{r,1}),\ h(v^m_{r,2}),\cdots,\ h(v^m_{r,n_r-1})$$

is strictly increasing.

Let $h(v^m_{r,n_r}) = h(v^m_{r,n_r-1})+1$ and $X = sum(h[V_r^m])$.

For $p \in \{1, 2,\cdots, m-1\}$ and $k \in \{1, 2,\cdots, r\}$, let $h(v^p_{k,n_k}) = X - sum(h[V_k^p \setminus \{v^p_{k,n_k}\}])$.

Thus $sum(h[V_k^p]) = h(v^p_{k,n_k}) + sum(h[V_k^p \setminus \{v^p_{k,n_k}\}] = X$.

Since the sequence

$$h[V_1^1 \setminus \{v^1_{1,n_1}\}],\ h[V_1^2 \setminus \{v^2_{1,n_1}\}],\cdots,\ h[V_1^m \setminus \{v^m_{1,n_1}\}],$$
$$h[V_2^1 \setminus \{v^1_{2,n_2}\}],\ h[V_2^2 \setminus \{v^2_{2,n_2}\}],\cdots,\ h[V_2^m \setminus \{v^m_{2,n_2}\}],$$
$$\vdots$$
$$h[V_r^1 \setminus \{v^1_{r,n_r}\}],\ h[V_r^2 \setminus \{v^2_{r,n_r}\}],\cdots,\ h[V_r^m \setminus \{v^m_{r,n_r}\}]$$

is strictly increasing, we have the sequence

$$h(v^m_{r,n_r}),\ h(v^{m-1}_{r,n_r}),\cdots,\ h(v^1_{r,n_r}),$$
$$h(v^m_{r-1,n_{r-1}}),\ h(v^{m-1}_{r-1,n_{r-1}}),\cdots,\ h(v^1_{r-1,n_{r-1}}),$$
$$\vdots$$
$$h(v^m_{1,n_1}),\ h(v^{m-1}_{1,n_1}),\cdots,\ h(v^1_{1,n_1})$$

is also strictly increasing. Thus $h(u)$ and $h(v)$ are pairwise distinct for all $u, v \in V(G)$. Hence h is one to one, that is $\left|h[V(G)]\right| = \left|V(G)\right|$.

Let $S = h[V(G)]$. Define $f : V(G) \to S$ as follows : $f(v) = h(v)$ for all $v \in V(G)$. Then f is a bijection and $sum(f[V_k^p]) = sum(h[V_k^p]) = X$ for all $k \in \{1, 2,\cdots, r\}$ and $p \in \{1, 2,\cdots, m\}$. By Lemma 2.1, f is an *S*-magic labeling of G. Therefore, G is S – magic with magic constant $X(r-1)$. ■

Example 2.4. Consider the graph $G = 3K_{3,4,4}$. Then G has order 33 and consists 3 components isomorphic with $K_{3,4,4}$. By Theorem 2.3, G is $S-$ magic (vertices are labeled as the Figure 1) with $S-$ magic constant 188.

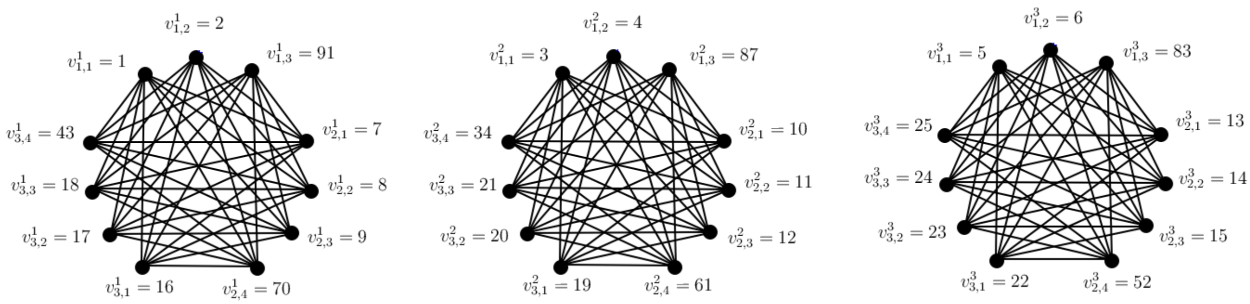

Figure 1 : The $S-$ magic labeling of $3K_{3,4,4}$ by using the labeling outlined in Theorem 2.3

The graph G adding an edge e is denoted by $G+e$ where e is not an edge of G. Godinho and Singh (Godinho and Singh, 2015) found a necessary condition for the existence of an $S-$ magic graph in Theorem 2.5. We use this to prove that $mK_{n_1,n_2,\ldots,n_r} + e$ is not an *S*-magic graph where $e = uv$ with u and v are vertices in the same partite set of $mK_{n_1,n_2,\ldots,n_r}$.

Theorem 2.5. (Godinho and Singh, 2015) If there exist two vertices u and v in a graph G such that

$$\left|(N_G(u)\setminus N_G(v)) \cup (N_G(v)\setminus N_G(u))\right| = 2$$

then G is not $S-$ magic.

Corollary 2.6. Let $G = mK_{n_1,n_2,\ldots,n_r}$ and $e = uv$ where u and v are vertices in the same partite of G. Then the graph $G+e$ is not $S-$ magic.

Proof. Assume that $H = G+e$. Note that $N_G(u) = N_G(v)$. Since $N_H(u) = N_G(u) \cup \{v\}$ and $N_H(v) = N_G(v) \cup \{u\}$, we have that $\left|(N_H(u)\setminus N_H(v)) \cup (N_H(v)\setminus N_H(u))\right| = 2$. By Theorem 2.5, H is not $S-$ magic. ■

Discussion

Godinho and Singh showed that the graph $K_{n_1,n_2,\ldots,n_r}$ where $n_1 \le n_2 \le \ldots \le n_r$ is $S-$magic if and only if $n_2 \ge 2$. Since if the graph $K_{n_1,n_2,\ldots,n_r}$ is $S-$magic, then $mK_{n_1,n_2,\ldots,n_r}$ may not be $S-$magic, for example, the graph $K_{1,2}$ is $S-$magic but $2K_{1,2}$ is not $S-$magic. In this paper, we show the condition for the existence of an $S-$magic labeling of the graph $mK_{n_1,n_2,\ldots,n_r}$, that is, the graph $mK_{n_1,n_2,\ldots,n_r}$ where $n_1 \le n_2 \le \ldots \le n_r$ and $m \ge 2$ is $S-$magic if and only if $n_1 \ge 2$.

Conclusions

For this research, we show a necessary and sufficient condition for the existence of an $S-$magic labelings of a union of disjoint complete multipartite graphs G, and show that the graph $G+uv$ is not $S-$magic if u and v are vertices in the same partite of G.

Acknowledgements

The authors are thankful to the Faculty of Science, Srinakharinwirot University for the financial support and we are also thankful to the referees for their helpful suggestions.

References

Arumugam, S., Froncek, D. and Kamatchi, N. (2011). Distance magic graphs – A survey. *Journal of the Indonesian Mathematical Society*, *Special Edition,* 11–16.

Gallian, J. A. (2017). A dynamic survey of graph labeling. *Electronic Journal of Combinatorics*, *20* #Ds6.

Godinho, A. and Singh, T. (2015). On *S*-magic graphs, *Electronic Notes in Discrete Mathematics*, *48*, 267–273.

Miller, M., Rodger, C. and Simanjuntak, R. (2003). Distance magic labelings of graphs, *Australasian Journal of Combinatorics*, *28*, 305–315.

Sankar, K., Sivakumaran, V., and Sethuraman, G. (2016). Distance magic labeling of join graph, *International Journal of Pure and Applied Mathematics*, *6*, 19–25.

Sugeng, K. A., Froncek, D., Miller, M., Ryan, J. and Walker, J. (2009). On Distance magic labelings of graphs, *Journal of Combinatorial Mathematics and Combinatorial Computing*, *71*, 39–48.

Vilfred, V. (1994). $\sum$-labelled graph and circulant graphs (Ph.D. Thesis). University of Kerala. Trivandrum. India.

West, D. B. (2001). Introduction to graph theory. Pretice Hall. New Jersey.