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กึ่งกรุปย่อยวิภัชนัยเฮซิแทนต์สหัชญาณนิยมของกึ่งกรุป

Intuitionistic Hesitant Fuzzy Subsemigroups of Semigroups

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บทคัดย่อ

จุดประสงค์ของงานวิจัยนี้คือต้องการขยายกึ่งกรุปย่อยวิภัชนัยเฮซิแทนต์ไปยังกึ่งกรุปย่อยวิภัชนัยเฮซิแทนต์สหัชญาณนิยมเช่นเดียวกับการขยายของกึ่งกรุปย่อยวิภัชนัยไปยังกึ่งกรุปย่อยวิภัชนัยสหัชญาณนิยมโดยการรวมกันของแนวคิดของเซตวิภัชนัยเฮซิแทนต์สหัชญาณนิยมและกึ่งกรุป ในงานนี้เราได้กำหนดนิยามของเซตวิภัชนัยเฮซิแทนต์สหัชญาณนิยมบนกึ่งกรุป และเราได้จำแนกลักษณะเฉพาะของกึ่งกรุปย่อย, ไอดีลซ้าย, ไอดีลขวา, ไอดีล, ไบ-ไอดีล, ไอดีลภายใน และ (1,2)-ไอดีล ในเทอมของกึ่งกรุปย่อยวิภัชนัยเฮซิแทนต์สหัชญาณนิยมของกึ่งกรุป และเราได้รับความสัมพันธ์ระหว่างกึ่งกรุปย่อยวิภัชนัยเฮซิแทนต์สหัชญาณนิยมและกึ่งกรุปย่อย

คำสำคัญ : เซตวิภัชนัยเฮซิแทนต์ ; เซตวิภัชนัยสหัชญาณนิยม ; กึ่งกรุปย่อยวิภัชนัยเฮซิแทนต์สหัชญาณนิยม

Abstract

The aim of this paper is to extend the hesitant fuzzy subsemigroups to intuitionistic hesitant fuzzy subsemigroups as well as extending of fuzzy subsemigroups to intuitionistic fuzzy subsemigroups by merging the concept of intuitionistic hesitant fuzzy set and subsemigroups. We define the definition of intuitionistic hesitant fuzzy sets on a semigroup and we characterize subsemigroups, left ideals, right ideals, ideals, bi-ideals, interior ideals and (1,2)-ideals in terms of intuitionistic hesitant fuzzy subsemigroups of semigroups. And relations between intuitionistic hesitant fuzzy subsemigroups and subsemigroups are obtained.

Keywords : hesitant fuzzy set ; intuitionistic fuzzy set ; intuitionistic hesitant fuzzy subsemigroup

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Introduction

The concept of an intuitionistic fuzzy set was introduced by Atanassov (Atanassov, 1986) which was defined as follows. An intuitionistic fuzzy set A on a nonempty set X is an object having the form $A := \{< x, (\mu(x), \gamma(x)) > | x \in X\}$ where $\mu : X \to [0,1]$, $\gamma : X \to [0,1]$ and $0 \leq \mu(x) + \gamma(x) \leq 1$ for all $x \in X$. This concept has been applied in different fields, for examples, logic programming, decision-making problem and medical diagnosis etc. Later, several researchers introduced the notion of intuitionistic fuzzy ideals in semigroups and investigated some related properties of ideals in semigroups and characterizations of intuitionistic fuzzy ideals in semigroups. The hesitant fuzzy set was introduced by Torra (Torra, 2010) which was a generalization of the fuzzy set and defined as follows. A hesitant fuzzy set on a nonempty set X is a function h from X to the power set of $[0,1]$ i.e. $h : X \to \wp([0,1])$. The hesitant fuzzy set theory has been applied to several practical problem, primarily in the area of decision making. Later, many researchers have applied this concept to semigroups. They introduced the notions of hesitant fuzzy subsemigroups, considered characterizations of hesitant fuzzy subsemigroups and investigated several properties.

Zadeh (Zadeh, 1965) introduced a concept of a fuzzy subset (fuzzy set) which is applied in many areas such as medical science, robotics, computer science, information science, control engineering, measure theory, logic, set theory, topology etc. After that, the concept of fuzzy algebraic systems was studied by Rosenfeld (Rosenfeld, 1971) for the first time. Kuroki (Kuroki, 1979) studied some properties of fuzzy subsemigroups of a semigroup and characterized a semigroup in terms of fuzzy subsets of semigroups. The concept of intuitionistic fuzzy sets was introduced by Atanassov (Atanassov, 1986) which is generalization of the notion of fuzzy sets. Kim and Jun (Kim and Jun, 2002) studied the intuitionistic fuzzy ideals in semigroup and investigated some properties. Torra (Torra, 2010) gave the concept of hesitant fuzzy sets which were generalization of fuzzy sets. Hesitant fuzzy set theory has been applied to several practical problems, primarily in the area of decision making. As you can see in (Rodriguez, Martinez & Herrera; Wei, 2012). Jun and Lee (Jun and Lee, 2015) applied notion of hesitant fuzzy sets to semigroup. They introduced the notions of hesitant fuzzy semigroups and hesitant fuzzy left (right) ideals, and investigated properties. They considered characterizations of hesitant fuzzy subsemigroups and hesitant fuzzy left (right) ideals. Beg and Rashid (Beg and Rashid, 2014) introduced the concept of Intuitionistic hesitant fuzzy sets by merging the concept of intuitionistic fuzzy sets and hesitant fuzzy sets. It helps to manage those situations of uncertainty in which some values are possible as membership values of element as well as non-membership values of the same element. However, both Beg and Rashid did not study algebraic properties of Intuitionistic hesitant fuzzy sets.

In this paper we extend the hesitant fuzzy subsemigroups to intuitionistic hesitant fuzzy subsemigroups as well as extending of fuzzy subsemigroups to intuitionistic fuzzy subsemigroups by merging the concept of intuitionistic hesitant fuzzy set and subsemigroups. We defined the definition of intuitionistic hesitant fuzzy sets on a semigroup and we characterize subsemigroups, left ideals, right ideals, ideals, bi-ideals, interior ideals and (1,2)-ideals in terms of intuitionistic hesitant fuzzy subsemigroups of semigroups. And relations between intuitionistic hesitant fuzzy subsemigroups and subsemigroups are discussed.

Methods

In this section, we will give basic definitions wich used in this paper.

Definition 1.1. A nonempty subset A of a semigroup S is called

(i) a *subsemigroup* of S if $AA \subseteq A$,

(ii) a *left (right) ideal* of S if $SA \subseteq A (AS \subseteq A)$,

(iii) an *ideal* of S if it is both a left and a right ideal of S,

(iv) a *bi-ideal* of S if it is both a subsemigroup of S and $ASA \subseteq A$,

(v) an *interior ideal* of S if it is both a subsemigroup of S and $SAS \subseteq A$,

(vi) a (1,2)-ideal of S if it is both a subsemigroup of S and $ASA^2 \subseteq A$.

We note that every ideal of a semigroup S is a subsemigroup, a bi-ideal, an interior ideal, a (1,2)-ideal of S and every bi-ideal of S is a (1,2)-ideal of S.

Zadeh (Zadeh, 1965) introduced a concept of a fuzzy subset (fuzzy set) which is applied in many areas. He gave the definition as follows:

Definition 1.2. (Zadeh, 1965) A *fuzzy subset* f of a nonempty set X is a function from X into unit closed interval $[0,1]$ of real numbers, i.e., $f : X \to [0,1]$.

The concept of intuitionistic fuzzy set was introduced by Atanassov (Atanassov, 1986), as a generalization of the notion of fuzzy set. He gave the definition as follows:

Definition 1.3. (Atanassov, 1986) An *intuitionistic fuzzy set* A on a nonempty set X is an object having the form

$$A := \{< x, \mu(x), \gamma(x) > | \, x \in X\}.$$

where the functions $\mu : X \to [0,1]$ and $\gamma : X \to [0,1]$ denote the degree of membership and the degree of non-membership of the element $x \in X$ respectively, and for every $x \in X$

$$0 \leq \mu(x) + \gamma(x) \leq 1.$$

The hesitant fuzzy set was introduced by Torra (Torra, 2010) which is a generalization of the fuzzy set and define as follows:

Definition 1.4. (Torra, 2010) Let X be a nonempty set. A *hesitant fuzzy set* on X is a function h from X to power set of $[0,1]$ i.e., $h: X \to \wp([0,1])$, where $\wp([0,1])$ denotes the power set of $[0,1]$.

We give an example of a hesitant fuzzy set on a semigroup.

Example 1.1. Note that $\mathbb{N}$ is a semigroup under usual multiplication. Define $h:\mathbb{N} \to \wp([0,1])$ by $h(n)=\{\frac{1}{n}\}$ for all $n \in \mathbb{N}$. Hence h is a hesitant fuzzy set of $\mathbb{N}$.

The definition of intuitionistic hesitant fuzzy sets introduced by Beg and Rashid (Beg and Rashid, 2014) and defined as follows:

Definition 1.5. (Beg and Rashid, 2014) Let μ and γ be functions from a set X to a power set of $[0,1]$. An *intuitionistic hesitant fuzzy set* on X is a set

$$A := \{< x, \mu(x), \gamma(x) > \mid x \in X\},$$

the values $\mu(x)$ and $\gamma(x)$ denote the membership values and non-membership values of the element $x \in X$ to the set A respectively, which satisfies

$$\max\{\mu(x)\} + \min\{\gamma(x)\} \leq 1,$$
$$\min\{\mu(x)\} + \max\{\gamma(x)\} \leq 1.$$

We note that the definition may not exists, when if values $\mu(x)$ or $\gamma(x)$ is an open interval in $[0,1]$. We also give an example as below.

Example 1.2. Note that $\mathbb{N}$ is a semigroup under usual multiplication. Define $\mu,\gamma : \mathbb{N} \to \wp([0,1])$ by

$$\mu(n) = (0, \frac{1}{n}) \text{ and } \gamma(n) = [0, \frac{1}{n+1}] \text{ for all } n \in \mathbb{N}.$$

Hence (μ, γ) is not an intuitionistic hesitant fuzzy set of $\mathbb{N}$, since $\mu(x)$ does not exists.

Results

From the definition 1.5 of intuitionistic hesitant fuzzy sets introduced by Beg, I. and Rashid, we note that the definition may not exists. Thus, in our work, we developed the definition of an intuitionistic hesitant fuzzy set as follows.

Definition 2.1. Let μ *and* γ be hesitant fuzzy sets of a set S. The set is called an *intuitionistic hesitant fuzzy set* (shortly, ihf set)

$$(\mu,\gamma):=\{<x,\mu(x),\gamma(x)>| x\in X\}$$

if for all $x\in S, s\in\mu(x), t\in\gamma(x)$,

$$0\leq s+t\leq 1.$$

Now, we present the following example according above definition.

Example 2.1. Note that $\mathbb{N}$ is a semigroup under usual multiplication. Define $\mu,\gamma:\mathbb{N}\to\wp([0,1])$ by

$$\mu(n)=[0,\frac{1}{2n}] \text{ and } \gamma(n)=(0,\frac{1}{3n+1}) \text{ for all } n\in\mathbb{N}.$$

Hence (μ,γ) is an ihf set of $\mathbb{N}$.

The following definitions are definition of intuitionistic hesitant fuzzy subsemigroups of S. Throughout this paper, we will denote a semigroup by S.

Definition 2.2. An intuitionistic hesitant fuzzy set (μ,γ) of S is called

(i) an *intuitionistic hesitant fuzzy subsemigroup* of S (shortly, ihf subsemigroup) if

$$\mu(x)\cap\mu(y)\subseteq\mu(xy) \text{ and } \gamma(xy)\subseteq\gamma(x)\cup\gamma(y) \text{ for all } x,y\in S.$$

(ii) an *intuitionistic hesitant fuzzy left ideal* of S (shortly, ihf *left ideal*) if

$$\mu(y)\subseteq\mu(xy) \text{ and } \gamma(xy)\subseteq\gamma(y) \text{ for all } x,y\in S.$$

(iii) an *intuitionistic hesitant fuzzy right ideal* of S (shortly, ihf *right ideal*) if

$$\mu(x)\subseteq\mu(xy) \text{ and } \gamma(xy)\subseteq\gamma(x) \text{ for all } x,y\in S.$$

(iv) an *intuitionistic hesitant fuzzy ideal* of S (shortly, ihf *ideal*) if

$$\mu(x)\cup\mu(y)\subseteq\mu(xy) \text{ and } \gamma(xy)\subseteq\gamma(x)\cap\gamma(y) \text{ for all } x,y\in S.$$

(v) an *intuitionistic hesitant fuzzy bi-ideal* of S (shortly, ihf *bi-ideal*) if it is an intuitionistic hesitant fuzzy subsemigroup of S and

$$\mu(x)\cap\mu(z)\subseteq\mu(xyz) \text{ and } \gamma(xyz)\subseteq\gamma(x)\cup\gamma(z) \text{ for all } x,y,z\in S.$$

(vi) an *intuitionistic hesitant fuzzy interior ideal* of S (shortly, ihf *interior ideal*) if it is an intuitionistic hesitant fuzzy subsemigroup of S and

$$\mu(a)\subseteq\mu(xay) \text{ and } \gamma(xay)\subseteq\gamma(a) \text{ for all } a,x,y\in S.$$

(vii) an *intuitionistic hesitant fuzzy (1,2)-ideal* of S (shortly, ihf *(1,2)-ideal*) if it is an intuitionistic hesitant fuzzy subsemigroup of S and

$$\mu(x)\cap\mu(y)\cap\mu(z)\subseteq\mu(xw(yz)) \text{ and } \gamma(xw(yz))\subseteq\gamma(x)\cup\gamma(y)\cup\gamma(z) \text{ for all } x,w,y,z\in S.$$

We note that every ihf ideal of S is an ihf subsemigroup of S, an ihf bi-ideal of S, an ihf interior ideal of S, an ihf (1,2)-ideal of S and every ihf bi-ideal of S is an ihf (1,2)-ideal of S.

Now, we give example of definitions of ihf subsemigroups as follow.

Example 2.2. Let $S=\{0,e,f,a,b\}$ be a set with the following Cayley table:

TABLE 1. Cayley table for the multiplication

•	0	e	f	a	b
0	0	0	0	0	0
e	0	e	0	a	0
f	0	0	f	0	b
a	0	a	0	0	e
b	0	0	b	f	0

Then S is a semigroup (Howie,1976). Define $\mu,\gamma : S \rightarrow \wp([0,1])$ by $\mu(0)=\mu(e)=\mu(f)=[0,1]$, $\mu(a)=\mu(b)=\{0\}$, $\gamma(0)=\gamma(e)=\gamma(f)=\{0\}$ and $\gamma(a)=\gamma(b)=[0,1]$. By the definition of ihf subsemigroup of *S,* we have (μ,γ) is an ihf subsemigroup of *S.*

Example 2.3. The (μ,γ) in Example 2.2. is an ihf interior ideal of *S.*

Example 2.4. Let $S=\{a,b,c,d,e\}$ be a set with the following Cayley table:

TABLE 2. Cayley table for the multiplication

•	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	a	a
c	a	a	c	c	e
d	a	a	c	d	e
e	a	a	c	c	e

Then S is a semigroup (Kim and Lee, 2005). Define $\mu,\gamma : S \longrightarrow \wp([0,1])$ by $\mu(a)=[0,0.6]$, $\mu(b)=[0,0.5]$, $\mu(c)=[0,0.4]$, $\mu(d)=\mu(e)=[0,0.3]$ and $\gamma(a)=\gamma(b)=[0,0.3]$, $\gamma(c)=[0,0.4]$, $\gamma(d)=[0,0.5]$, $\gamma(e)=[0,0.6]$. By the definition of ihf bi-ideal of S, we have (μ,γ) is an ihf bi-ideal of S.

Example2.5 Let $S=\{a,b,c,d\}$ be a set with the following Cayley table:

TABLE 3. Cayley table for the multiplication

•	a	b	c	d
a	a	a	a	a
b	a	b	c	a
c	a	c	c	b
d	a	b	d	d

Then S is a semigroup. (Wang, 2019) Define $\mu,\gamma : S \longrightarrow \wp([0,1])$ by $\mu(a)=[0,0.9]$, $\mu(b)=[0,0.7]$, $\mu(c)=[0,0.5]$, $\mu(d)=[0,0.4]$ and $\gamma(a)=\{0\}$, $\gamma(b)=[0,0.1]$, $\gamma(c)=[0,0.3]$, $\gamma(d)=[0,0.4]$. By the definition of ihf (1,2)-ideal of S, we have (μ,γ) is an ihf (1,2)-ideal of S.

Definition 2.3. Let A be a nonempty subset of S, the *characteristic intuitionistic hesitant fuzzy set* (shortly, cihf set) and is denoted by $\tilde{\chi}_A$,

$$\tilde{\chi}_A = \{< x, \chi_A, \chi_{A^c} > \mid x \in S\},$$

where χ_A and χ_{A^c} are hesitant fuzzy subsets defined as follows:

$$\chi_A : S \to \wp([0,1]), x \mapsto \begin{cases} [0,1] & \text{if } x \in A, \\ \varnothing & \text{if } x \notin A, \end{cases}$$

and

$$\chi_{A^c} : S \to \wp([0,1]), x \mapsto \begin{cases} \varnothing & \text{if } x \in A, \\ [0,1] & \text{if } x \notin A. \end{cases}$$

The following lemma we characterize subsemigroups in terms of characteristic intuitionistic hesitant fuzzy subsemigroups of semigroups.

Lemma 2.1. Let A be a nonempty subset of S. Then A is a subsemigroup of S if and only if the cihf set $\tilde{\chi}_A = (\chi_A, \chi_{A^c})$ is an ihf subsemigroup of S.

Proof. ($\Longrightarrow$) Assume that A is a subsemigroup of S. Let $x, y \in S$. We consider the following two cases.

Case 1. Suppose $xy \notin A$. Then $\chi_A(xy) = \varnothing$, $\chi_{A^c}(xy) = [0,1]$. Since A is a subsemigroup of S and $xy \notin A$, we have $x \notin A$ or $y \notin A$ which implies that $\chi_A(x) \cap \chi_A(y) = \varnothing$ *and* $\chi_{A^c}(x) \cup \chi_{A^c}(y) = [0,1]$. Thus

$$\chi_A(x) \cap \chi_A(y) = \varnothing = \chi_A(xy) \text{ and } \chi_{A^c}(xy) = [0,1] = \chi_{A^c}(x) \cup \chi_{A^c}(y).$$

Case 2. Suppose $xy \in A$. Then $\chi_A(xy) = [0,1]$, $\chi_{A^c}(xy) = \varnothing$. Thus

$$\chi_A(x) \cap \chi_A(y) \subseteq [0,1] = \chi_A(xy) \text{ and } \chi_{A^c}(xy) = \varnothing \subseteq \chi_{A^c}(x) \cup \chi_{A^c}(y).$$

Hence, by the two cases, we have $\tilde{\chi}_A$ is an ihf subsemigroup of S.

($\Longleftarrow$) Assume that $\tilde{\chi}_A$ is an ihf subsemigroup of S. Let $x, y \in A$. Then $\chi_A(x) = \chi_A(y) = [0,1]$. Since $\tilde{\chi}_A$ is an ihf subsemigroup of S, we have

$$[0,1] = \chi_A(x) \cap \chi_A(y) \subseteq \chi_A(xy) .$$

Thus $xy \in A$. Hence A is a subsemigroup of S.

The following theorems we characterize several ideals in terms of several cihf ideals of semigroups.

Theorem 2.1. Let A be a nonempty subset of S. Then A is a left ideal (right ideal, ideal) of S if and only if the cihf set $\tilde{\chi}_A = (\chi_A, \chi_{A^c})$ of A is an ihf left ideal (right ideal, ideal) of S.

Proof. ($\Longrightarrow$) Assume that A is a left ideal of S. Let $x, y \in S$. If $y \notin A$, then $\chi_A(y) = \varnothing$ and $\chi_{A^c}(y) = [0,1]$ which implies that $\chi_A(y) \subseteq \chi_A(xy)$ and $\chi_{A^c}(xy) \subseteq \chi_{A^c}(y)$.

On the other hand, suppose that $y \in A$. Then $\chi_A(y) = [0,1]$ and $\chi_{A^c}(y) = \varnothing$. Since A is a left ideal of S, we have $xy \in A$. Thus

$$\chi_A(xy) = [0,1] \text{ and } \chi_{A^c}(xy) = \varnothing .$$

Hence

$$\chi_A(y) \subseteq \chi_A(xy) \text{ and } \chi_{A^c}(xy) \subseteq \chi_{A^c}(y) .$$

Therefore $\tilde{\chi}_A$ is an ihf left ideal of S.

($\Longleftarrow$) Assume that $\tilde{\chi}_A$ is an ihf left ideal of S. Let $x \in S$ and $y \in A$. Then $\chi_A(y) = [0,1]$. Since $\tilde{\chi}_A$ is an ihf left ideal of S, we have $[0,1] = \chi_A(y) \subseteq \chi_A(xy)$. Therefore $xy \in A$. Hence A is a left ideal of S.

Theorem 2.2. Let B be a nonempty subset of S. Then B is a bi-ideal of S if and only if the cihf set $\tilde{\chi}_B = (\chi_B, \chi_{B^c})$ of B is an ihf bi-ideal of S.

Proof. ($\Longrightarrow$) Assume that B is a bi-ideal of S. Since B is a subsemigroup of S, we have $\tilde{\chi}_B$ is an ihf subsemigroup of S, by Lemma 2.1. Let $x,y,z\in S$. Suppose that $x\notin B$ or $z\notin B$. Then $\chi_B(x)\cap\chi_B(z)=\varnothing$ and $\chi_{B^c}(x)\cup\chi_{B^c}(z)=[0,1]$. Then

$$\chi_B(x)\cap\chi_B(z)\subseteq\chi_B(xyz) \text{ and } \chi_{B^c}(xyz)\subseteq\chi_{B^c}(x)\cup\chi_{B^c}(z).$$

On the other hand, suppose that $x,z\in B$, then $\chi_B(x)=\chi_B(z)=[0,1]$ and $\chi_{B^c}(x)=\chi_{B^c}(Z)=\varnothing$. Since B is a bi-ideal of S, we have $xyz\in B$. Thus $\chi_B(xyz)=[0,1]$ and $\chi_{B^c}(xyz)=\varnothing$. Hence

$$\chi_B(x)\cap\chi_B(z)\subseteq\chi_B(xyz) \text{ and } \chi_{B^c}(xyz)\subseteq\chi_{B^c}(x)\cup\chi_{B^c}(z).$$

Therefore $\tilde{\chi}_B$ is an ihf bi-ideal of S.

($\Longleftarrow$) Assume that $\tilde{\chi}_B$ is an ihf bi-ideal of S. Then B is a subsemigroup of S, by Lemma 2.5. Let $x,z\in B$ and $y\in S$, then $\chi_B(x)=\chi_B(z)=[0,1]$. Since $\tilde{\chi}_B$ is an ihf bi- ideal of S, we have $\chi_B(x)\cap\chi_B(z)\subseteq\chi_B(xyz)$. Thus

$$[0,1]=\chi_B(x)\cap\chi_B(z)\subseteq\chi_B(xyz).$$

Hence $xyz\in B$. Therefore B is a bi-ideal of S.

Theorem 2.3. Let A be a nonempty subset of S. Then A is an interior ideal of S if and only if the cihf set $\tilde{\chi}_A=(\chi_A,\chi_{A^c})$ of A is an ihf interior ideal of S.

Proof. ($\Longrightarrow$) Assume that A is an interior ideal of S. By Lemma 2.1, we have $\tilde{\chi}_A$ is an ihf subsemigroup of S. Let $a,x,y\in S$. If $a\notin A$, then $\chi_A(a)=\varnothing\subseteq\chi_A(xay)$ and $\chi_{A^c}(xay)\subseteq[0,1]=\chi_{A^c}(a)$. On the other hand, if $a\in A$. Since A is an interior ideal of S, we have

$$\chi_A(a)=[0,1]=\chi_A(xay) \text{ and } \chi_{A^c}(xay)=\varnothing=\chi_{A^c}(a).$$

Therefore $\tilde{\chi}_A$ is an ihf interior ideal of S.

($\Longleftarrow$) Assume that $\tilde{\chi}_A$ is an ihf interior ideal of S. By Lemma 2.5, we have A is a subsemigroup of S. Let $x,y\in S$ and $a\in A$. Then $[0,1]=\chi_A(a)\subseteq\chi_A(xay)$ and so $xay\in A$. Hence A is an interior ideal of S.

Theorem 2.4. Let A be a nonempty subset of S. Then A is a $(1,2)$-ideal of S if and only if the cihf set $\tilde{\chi}_A=(\chi_A,\chi_{A^c})$ of A is an ihf (1,2)-ideal of S.

Proof. ($\Longrightarrow$) Assume that A is a (1,2)-ideal of S. By Lemma 2.1, we have $\tilde{\chi}_A$ is an ihf subsemigroup of S. Let $w,x,y,z\in S$. If x or y or $z\notin A$. Then

$$\chi_A(xw(yz))\supseteq\varnothing=\chi_A(x)\cap\chi_A(y)\cap\chi_A(z) \text{ and } \chi_{A^c}(xw(yz))\subseteq[0,1]=\chi_{A^c}(x)\cup\chi_{A^c}(y)\cup\chi_{A^c}(z).$$

If $x,y,z\in A$, then we have

$$\chi_A(xw(yz))=[0,1]\supseteq\chi_A(x)\cap\chi_A(y)\cap\chi_A(z) \text{ and } \chi_{A^c}(xw(yz))=\varnothing\subseteq\chi_{A^c}(x)\cup\chi_{A^c}(y)\cup\chi_{A^c}(z).$$

Hence $\tilde{\chi}_A$ is an ihf (1,2)-ideal of S.

($\Leftarrow$) Assume that $\tilde{\chi}_A$ is an ihf (1,2)-ideal of S. By Lemma 2.5, we have A is a subsemigroup of S. Let $w \in S$ and $w, x, y, z \in A$. Then $\chi_A(xw(yz)) \supseteq \chi_A(x) \cap \chi_A(y) \cap \chi_A(z) = [0,1]$. Hence $xw(yz) \in A$. Therefore A is a (1,2)-ideal of S.

Now, we give the definition of the hesitant $(\alpha,\beta)-cut$.

Definition 2.4. *Let* μ *and* γ be hesitant fuzzy sets on a nonempty set S and let $\alpha, \beta \in \wp([0,1])$.

The set

$$U(\mu;\alpha) = \{x \in S \mid \mu(x) \supseteq \alpha\} \text{ and } L(\gamma;\beta) = \{x \in S \mid \gamma(x) \subseteq \beta\}.$$

The set

$$C_{(\mu,\gamma;\alpha,\beta)} = U(\mu;\alpha) \cap L(\gamma;\beta).$$

is called the *hesitant* $(\alpha,\beta)-cut$.

The following definition are definition of hesitant fuzzy subsemigroup and anti-hesitant fuzzy subsemigroup of S.

Definition 2.4. (Jun and Lee, 2015) A hesitant fuzzy set h of S is called a *hesitant fuzzy subsemigroup* of S if

$$h(x) \cap h(y) \subseteq h(xy) \text{ for all } x, y \in S.$$

Definition 2.5. A hesitant fuzzy set h of S is called an *anti-hesitant fuzzy subsemigroup* of S if

$$h(xy) \subseteq h(x) \cup h(y) \text{ for all } x, y \in S.$$

Remark 2.1. Intersection of subsemigroups of S is either an empty set or a subsemigroup of S.

Remark 2.2. Let S be a semigroup and define the hesitant fuzzy set O of S by $\mathrm{O}(x) = \{0\}$ for all $x \in S$. The following statements are true.

(1) If (μ,γ) is an ihf subsemigroup of a semigroup S, then μ is a hesitant fuzzy subsemigroup and γ is an anti-hesitant fuzzy subsemigroup of S. However, the converse is not true.

(2) A hesitant fuzzy set μ of S is a hesitant fuzzy subsemigroup of S if and only if (μ,O) is an ihf subsemigroup of S.

(3) A hesitant fuzzy set γ of S is an anti-hesitant fuzzy subsemigroup of S if and only if (O,γ) is an ihf subsemigroup of S.

The following lemmas and theorems are shows relations between intuitionistic hesitant fuzzy subsemigroups and subsemigroups by the hesitant $(\alpha,\beta)-cut$ of (μ,γ).

Lemma 2.2. If μ is a hesitant fuzzy subsemigroup and γ is an anti-hesitant fuzzy subsemigroup of S then the following statements hold.

(1) $L(\gamma;\beta)$ is either an empty set or a subsemigroup of S for all $\beta \in \wp([0,1])$,

(2) $U(\mu;\alpha)$ is either an empty set or a subsemigroup of S for all $\alpha \in \wp([0,1])$,

(3) $C_{(\mu,\gamma;\alpha,\beta)}$ is either an empty set or a subsemigroup of S for all $\alpha,\beta \in \wp([0,1])$.

Proof. Assume that μ is a hesitant fuzzy subsemigroup and γ is an anti-hesitant fuzzy subsemigroup of S.

(1) Let $\beta \in \wp([0,1])$ and let $x,y \in L(\gamma;\beta)$. Then $\gamma(x) \subseteq \beta$ and $\gamma(y) \subseteq \beta$. By the assumption, we have that $\gamma(xy) \subseteq \gamma(x) \cup \gamma(y) \subseteq \beta$. Hence $xy \in L(\gamma;\beta)$. Therefore $L(\gamma;\beta)$ is a subsemigroup of S.

(2) The proof is similar to the proof of (1).

(3) It follows from (1), (2) and Remark 2.1.

Theorem 2.5. Let (μ,γ) be an ihf set of S. Then (μ,γ) is an ihf subsemigroup of S if and only if $C_{(\mu,\gamma;\alpha,\beta)}$ is empty or a subsemigroup of S for all $\alpha,\beta \in \wp([0,1])$.

Proof. ($\Longrightarrow$) It follows from Remark 2.2 and Lemma 2.13.

($\Longleftarrow$) Assume that $C_{(\mu,\gamma;\alpha,\beta)}$ is empty or a subsemigroup of S for all $\alpha,\beta \in \wp([0,1])$. Let $x,y \in S$. Then $\mu(x) \cap \mu(y) \in \wp([0,1])$ and $\gamma(x) \cup \gamma(y) \in \wp([0,1])$. Thus $x,y \in C_{(\mu,\gamma;\mu(x)\cap\mu(y),\gamma(x)\cup\gamma(y))}$. By the assumption, we get $C_{(\mu,\gamma;\mu(x)\cap\mu(y),\gamma(x)\cup\gamma(y))}$ is a subsemigroup of S and so $xy \in C_{(\mu,\gamma;\mu(x)\cap\mu(y),\gamma(x)\cup\gamma(y))}$. Hence $\mu(x) \cap \mu(y) \subseteq \mu(xy)$ and $\gamma(xy) \subseteq \gamma(x) \cup \gamma(y)$. Therefore (μ,γ) is an ihf subsemigroup of S.

Definition 2.6. (Jun and Lee, 2015) Let h be a hesitant fuzzy set of S is called a *hesitant fuzzy left ideal (right ideal)* of S if $h(y) \subseteq h(xy)$ $(h(x) \subseteq h(xy))$ for all $x,y \in S$ and it is called a *hesitant fuzzy ideal* of S if $h(x) \cup h(y) \subseteq h(xy)$ for all $x,y \in S$.

Definition 2.7. A hesitant fuzzy set h of S is called an *anti-hesitant fuzzy left ideal (right ideal)* of S if $h(xy) \subseteq h(y)$ $(h(xy) \subseteq h(x))$ for all $x,y \in S$ and is called an *anti-hesitant fuzzy ideal* of S if $h(xy) \subseteq h(x) \cap h(y)$ for all $x,y \in S$.

Remark 2.3. Intersection of left ideals (right ideals, ideals) of S is either an empty set or a left ideal (right ideal, ideal) of S.

Remark 2.4. Let S be a semigroup and define the hesitant fuzzy set O of S by $\mathrm{O}(x) = \{0\}$ for all $x \in S$. The following statements are true.

(1) If (μ,γ) is an ihf left ideal (right ideal, ideal) of S, then μ is a hesitant fuzzy left ideal (right ideal, ideal) and γ is an anti-hesitant fuzzy left ideal (right ideal, ideal) of S. However, the converse is not true.

(2) A hesitant fuzzy set μ of S is a hesitant fuzzy left ideal (right ideal, ideal) of S if and only if (μ,O) is an ihf left ideal (right ideal, ideal) of S.

(3) A hesitant fuzzy set γ of S is an anti-hesitant fuzzy left ideal (right ideal, ideal) of S if and only if (O,γ) is an ihf left ideal (right ideal, ideal) of S.

Lemma 2.3. If μ is a hesitant fuzzy left ideal (right ideal, ideal) and γ is an anti-hesitant fuzzy left ideal (right ideal, ideal) of S then the following statements hold.

(1) $L(\gamma;\beta)$ is either an empty set or a left ideal (right ideal, ideal) of S for all $\beta \in \wp([0,1])$.

(2) $U(\mu;\alpha)$ is either an empty set or a left ideal (right ideal, ideal) of S for all $\alpha \in \wp([0,1])$.

(3) $C_{(\mu,\gamma;\alpha,\beta)}$ is either an empty set or a left ideal (right ideal, ideal) of S for all $\alpha,\beta \in \wp([0,1])$.

Proof. Assume that μ is a hesitant fuzzy left ideal and γ is an anti-hesitant fuzzy left ideal of S.

(1) Let $\alpha,\beta \in \wp([0,1])$ and let $x \in S$ and $y \in L(\gamma;\beta)$. The $\gamma(y) \subseteq \beta$. By the assumption, we have that $\gamma(xy) \subseteq \gamma(y) \subseteq \beta$. Hence $xy \in L(\gamma;\beta)$. Therefore $L(\gamma;\beta)$ is a left ideal of S.

(2) The proof is similar to the proof of (1).

(3) It follows from (1), (2) and Remark 2.3.

Theorem 2.6. Let (μ,γ) be an ihf set of S. Then (μ,γ) is an ihf left ideal(right ideal, ideal) of S if and only if $C_{(\mu,\gamma;\alpha,\beta)}$ is a left ideal (right ideal, ideal) of S for all $\alpha,\beta \in \wp[0,1])$.

Proof. $(\Longrightarrow)$ It follows from Remark 2.4 and Lemma 2.17.

$(\Longleftarrow)$ Assume that $C_{(\mu,\gamma;\alpha,\beta)}$ is a left ideal of S. Let $x,y \in S$. Then $\mu(y),\gamma(y) \in \wp([0,1])$. Thus $y \in C_{(\mu,\gamma;\mu(y),\gamma(y))}$. By assumption, we have $xy \in C_{(\mu,\gamma;\mu(y),\gamma(y))}$. Hence $\mu(y) \subseteq \mu(xy)$ and $\gamma(xy) \subseteq \gamma(y)$. Therefore (μ,γ) is an ihf left ideal of S.

Definition 2.8. (Jun and Lee, 2015) A hesitant fuzzy set h of S is called a *hesitant fuzzy bi-ideal* of S if it is a hesitant fuzzy subsemigroup of S and $h(x) \cap h(z) \subseteq h(xyz)$ for all $x,y,z \in S$.

Definition 2.9. A hesitant fuzzy set h of S is called an *anti-hesitant fuzzy bi-ideal* of S if it is an anti- hesitant fuzzy subsemigroup of S and $h(xyz) \subseteq h(x) \cup h(z)$ for all $x,y,z \in S$.

Remark 2.5. Intersection of bi-ideal of S is either an empty set or a bi-ideal of S.

Remark 2.6. Let S be a semigroup and define the hesitant fuzzy set O of S by $\mathrm{O}(x)=\{0\}$ for all $x\in S$. The following statements are true.

(1) If (μ,γ) is an ihf bi-ideal of S, then μ is a hesitant fuzzy bi-ideal and γ is an anti-hesitant fuzzy bi-ideal of S. However, the converse is not true.

(2) A hesitant fuzzy set μ of S is a hesitant fuzzy bi-ideal of S if and only if (μ,O) is an ihf bi-ideal of S.

(3) A hesitant fuzzy set γ of S is an anti-hesitant fuzzy bi-ideal of S if and only if (O,γ) is an ihf bi-ideal of S.

Lemma 2.4. If μ is a hesitant fuzzy bi-ideal and γ is an anti-hesitant fuzzy bi-ideal of S then the following statements hold.

(1) $L(\gamma;\beta)$ is either an empty set or a bi-ideal of S for all $\beta\in\wp([0,1])$.

(2) $U(\mu;\alpha)$ is either an empty set or a bi-ideal of S for all $\alpha\in\wp([0,1])$.

(3) $C_{(\mu,\gamma;\alpha,\beta)}$ is either an empty set or a bi-ideal of S for all $\alpha,\beta\in\wp([0,1])$.

Proof. Assume that μ is a hesitant fuzzy bi-ideal and γ is an anti-hesitant fuzzy bi-ideal of S. Since μ is a hesitant fuzzy subsemigroups and γ is an anti-hesitant fuzzy subsemigroups of S, we have that $U(\mu;\alpha)$ and $L(\gamma;\beta)$ are subsemigroups of S, by Lemma 2.11.

(1) Let $\alpha,\beta\in\wp([0,1])$ and let $y\in S$ and $x,z\in L(\gamma;\beta)$. Then $\gamma(x)\subseteq\beta$, $\gamma(z)\subseteq\beta$. By the assumption, we have $\gamma(xyz)\subseteq\gamma(x)\cup\gamma(z)\subseteq\beta$. Hence $xyz\in L(\gamma;\beta)$. Therefore $L(\gamma;\beta)$ is a bi-ideals of S.

(2) The proof is similar to the proof of (1).

(3) It follows from (1), (2) and Remark 2.5.

Theorem 2.7. Let (μ,γ) be an ihf set of S. Then (μ,γ) is an ihf bi-ideal of S if and only if $C_{(\mu,\gamma;\alpha,\beta)}$ is a bi-ideal of S for all $\alpha,\beta\in\wp([0,1])$.

Proof. ($\Longrightarrow$) It follows from Remark 2.6 and Lemma 2.4.

($\Longleftarrow$) Assume that $C_{(\mu,\gamma;\alpha,\beta)}$ is a bi-ideal of S. Since $C_{(\mu,\gamma;\alpha,\beta)}$ is a subsemigroup of S, we have (μ,γ) is an ihf subsemigroup of S by Theorem 2.12. Let $x,y,z\in S$. Then $\mu(x)\cap\mu(z),\gamma(x)\cup\gamma(z)\in\wp([0,1])$. Thus $x,z\in C_{(\mu,\gamma;\mu(x)\cap\mu(z),\gamma(x)\cup\gamma(z))}$. By assumption, we have $xyz\in C_{(\mu,\gamma;\mu(x)\cap\mu(z),\gamma(x)\cup\gamma(z))}$. Hence $\mu(x)\cap\mu(z)\subseteq\mu(xyz)$ and $\gamma(xyz)\subseteq\gamma(x)\cup\gamma(z)$. Therefore (μ,γ) is an ihf bi-ideal of S.

Definition 2.10. (Jun and Lee, 2015) A hesitant fuzzy set h of S is called a *hesitant fuzzy interior ideal* of S if it is a hesitant fuzzy subsemigroup of S and $h(a) \subseteq h(xay)$ for all $a, x, y \in S$.

Definition 2.11. A hesitant fuzzy set h of S is called an *anti-hesitant fuzzy interior ideal* of S if it is an anti- hesitant fuzzy subsemigroup of S and $h(xay) \subseteq h(a)$ for all $a, x, y \in S$.

Remark 2.7. Intersection of interior ideal of S is either an empty set or an interior ideal of S.

Remark 2.8. Let S be a semigroup and define the hesitant fuzzy set O of S by $\mathrm{O}(x) = \{0\}$ for all $x \in S$. The following statements are true.

(1) If (μ, γ) is an ihf interior ideal of S, then μ is a hesitant fuzzy interior ideal and γ is an anti-hesitant fuzzy interior ideal of S. However, the converse is not true.

(2) A hesitant fuzzy set μ of S is a hesitant fuzzy interior ideal of S if and only if (μ, O) is an ihf interior ideal of S.

(3) A hesitant fuzzy set γ of S is an anti-hesitant fuzzy interior ideal of S if and only if (O, γ) is an ihf interior ideal of S.

Lemma 2.5. If μ is a hesitant fuzzy interior ideal and γ is an anti-hesitant fuzzy interior ideal of S then the following statements hold.

(1) $L(\gamma; \beta)$ is either an empty set or an interior ideal of S for all $\beta \in \wp([0,1])$.

(2) $U(\mu; \alpha)$ is either an empty set or an interior ideal of S for all $\alpha \in \wp([0,1])$.

(3) $C_{(\mu,\gamma;\alpha,\beta)}$ is either an empty set or an interior ideal of S for all $\alpha, \beta \in \wp([0,1])$.

Proof. Assume that μ is a hesitant fuzzy interior ideal and γ is an anti-hesitant fuzzy interior ideal of S. Since μ is a hesitant fuzzy subsemigroups and γ is an anti-hesitant fuzzy subsemigroups of S, we have that $U(\mu; \alpha)$ and $L(\gamma; \beta)$ are subsemigroups of S, by Lemma 2.2.

(1) Let $\alpha, \beta \in \wp([0,1])$ and let $x, y \in S$ and $a \in L(\gamma; \beta)$. Then $\gamma(a) \subseteq \beta$. By the assumption, we have $\gamma(xay) \subseteq \gamma(a) \subseteq \beta$. Hence $xay \in L(\gamma; \beta)$. Therefore $L(\gamma; \beta)$ is an interior ideal of S.

(2) The proof is similar to the proof of (1).

(3) It follows from (1), (2) and Remark 2.7.

Theorem 2.8. Let (μ, γ) be an ihf set of S. Then (μ, γ) is an ihf interior ideal of S if and only if $C_{(\mu,\gamma;\alpha,\beta)}$ is an interior ideal of S for all $\alpha, \beta \in \wp([0,1])$.

Proof. $(\Longrightarrow)$ It follows from Remark 2.8 and Lemma 2.5.

($\Leftarrow$) Assume that $C_{(\mu,\gamma;\alpha,\beta)}$ is an interior ideal of S. Since $C_{(\mu,\gamma;\alpha,\beta)}$ is a subsemigroup of S, we have (μ,γ) is an ihf subsemigroup of S by Theorem 2.12. Let $a,x,y \in S$. Then $\mu(a),\gamma(a) \in \wp([0,1])$. Thus $a \in C_{(\mu,\gamma;\mu(a),\gamma(a))}$. By assumption, we have $xay \in C_{(\mu,\gamma;\mu(a),\gamma(a))}$. Hence $\mu(a) \subseteq \mu(xay)$ and $\gamma(xay) \subseteq \gamma(a)$. Therefore (μ,γ) is an ihf interior ideal of S.

Definition 2.12. (Jun and Lee, 2015) A hesitant fuzzy set h of S is called a *hesitant fuzzy (1,2)-ideal* of S if it is a hesitant fuzzy subsemigroup of S and $h(x) \cap h(y) \cap h(z) \subseteq h(xw(yz))$ for all $w,x,y,z \in S$.

Definition 2.13. A hesitant fuzzy set h of S is called an *anti-hesitant fuzzy (1,2)-ideal* of S if it is an anti- hesitant fuzzy subsemigroup of S and $h(xw(yz)) \subseteq h(x) \cup h(y) \cup h(z)$ for all $w,x,y,z \in S$.

Remark 2.9. Intersection of (1,2)-ideal of S is either an empty set or a (1,2)-ideal of S.

Remark 2.10. Let S be a semigroup and define the hesitant fuzzy set O of S by $O(x) = \{0\}$ for all $x \in S$. The following statements are true.

(1) If (μ,γ) is an ihf (1,2)-ideal of S, then μ is a hesitant fuzzy (1,2)-ideal and γ is an anti-hesitant fuzzy (1,2)-ideal of S. However, the converse is not true.

(2) A hesitant fuzzy set μ of S is a hesitant fuzzy (1,2)-ideal of S if and only if (μ, O) is an ihf (1,2)-ideal of S.

(3) A hesitant fuzzy set γ of S is an anti-hesitant fuzzy (1,2)-ideal of S if and only if (O,γ) is an ihf (1,2)-ideal of S.

Lemma 2.6. If μ is a hesitant fuzzy (1,2)-ideal and γ is an anti-hesitant fuzzy (1,2)-ideal of S then the following statements hold.

(1) $L(\gamma;\beta)$ is either an empty set or a (1,2)-ideal of S for all $\beta \in \wp([0,1])$.

(2) $U(\mu;\alpha)$ is either an empty set or a (1,2)-ideal of S for all $\alpha \in \wp([0,1])$.

(3) $C_{(\mu,\gamma;\alpha,\beta)}$ is either an empty set or a (1,2)-ideal of S for all $\alpha,\beta \in \wp([0,1])$.

Proof. Assume that μ is a hesitant fuzzy (1,2)-ideal and γ is an anti-hesitant fuzzy (1,2)-ideal of S. Since μ is a hesitant fuzzy subsemigroups and γ is an anti-hesitant fuzzy subsemigroups of S, we have that $U(\mu;\alpha)$ and $L(\gamma;\beta)$ are subsemigroups of S by Lemma 2.2.

(1) Let $\alpha,\beta \in \wp([0,1])$ and let $w \in S$, $x,y,z \in L(\gamma;\beta)$. Then $\gamma(x) \cup \gamma(y) \cup \gamma(z) \subseteq \beta$. By the assumption, we have $\gamma(xw(yz)) \subseteq \gamma(x) \cup \gamma(y) \cup \gamma(z) \subseteq \beta$. Hence $xw(yz) \in L(\gamma;\beta)$. Therefore $L(\gamma;\beta)$ is a (1,2)-ideals of S.

(2) The proof is similar to the proof of (1).

(3) It follows from (1), (2) and Remark 2.9.

Theorem 2.9. Let (μ,γ) be an ihf set of S. Then (μ,γ) is an ihf (1,2)-ideal of S if and only if $C_{(\mu,\gamma;\alpha,\beta)}$ is a (1,2)-ideal of S for all $\alpha,\beta \in \wp([0,1])$.

Proof. $(\Longrightarrow)$ It follows from Remark 2.10 and Lemma 2.29.

$(\Longleftarrow)$ Assume that $C_{(\mu,\gamma;\alpha,\beta)}$ is a (1,2)-ideal of S. Since $C_{(\mu,\gamma;\alpha,\beta)}$ is a subsemigroup of S we have (μ,γ) is an ihf subsemigroup of S, by Theorem 2.12. Let $w,x,y,z \in S$.

Then $\mu(x)\cap\mu(y)\cap\mu(z),\gamma(x)\cup\gamma(y)\cup\gamma(z) \in \wp([0,1])$. Thus

$$x,y,z \in C_{(\mu,\gamma;\mu(x)\cap\mu(y)\cap\mu(z),\gamma(x)\cup\gamma(y)\cup\gamma(z))}.$$

By assumption, we have $xw(yz) \in C_{(\mu,\gamma;\mu(x)\cap\mu(y)\cap\mu(z),\gamma(x)\cup\gamma(y)\cup\gamma(z))}$. Hence

$$\mu(x)\cap\mu(y)\cap\mu(z) \subseteq \mu(xw(yz)) \text{ and } \gamma(xw(yz)) \subseteq \gamma(x)\cup\gamma(y)\cup\gamma(z).$$

Therefore (μ,γ) is an ihf (1,2)-ideal of S.

Discussion

The aim of this paper extended the concept of the hesitant fuzzy subsemigroups to intuitionistic hesitant fuzzy subsemigroups by merging the concept of intuitionistic hesitant fuzzy set and subsemigroups. Our purpose is to introduced the concept of intuitionistic hesitant fuzzy set in semigroup, characterized subsemigroups and several ideals and investigated relations between intuitionistic hesitant fuzzy subsemigroups and subsemigroups as follows. Firstly, we defined an intuitionistic hesitant fuzzy set on a semigroup which was similar to the concept of an intuitionistic fuzzy set on a semigroup. The point of view to note here was the definition of intuitionistic hesitant fuzzy sets introduced by Beg and Rashid (Beg and Rashid, 2014) (see Definition 1.5). We note that the definition may not exists, if values $\mu(x)$ or $\gamma(x)$ is an open interval in $[0,1]$. Moreover, we give an Example 1.2, which not support the definition 1.5. Thus, in our work, we developed the definition of an intuitionistic hesitant fuzzy set for any values $\mu(x)$ and $\gamma(x)$ as the Definition 2.1. Therefore, it could show that our works were generalizations of the concept of intuitionistic hesitant fuzzy sets introduced by Beg and Rashid. Secondly, we characterized subsemigroups, left ideals, right ideals, ideals, bi-ideals, interior ideals and (1,2)-ideals in terms of intuitionistic hesitant fuzzy subsemigroups of semigroups. These could be described that characterization of susemigroups could consider via special functions, that was characteristic intuitionistic hesitant fuzzy set. Furthermore, we investigated relations between intuitionistic hesitant fuzzy subsemigroups and hesitant (α,β)- cut.

Conclusions

In this paper, we defined the definition of intuitionistic hesitant fuzzy sets on a semigroup and we characterized subsemigroups shuch as left ideal, right ideal, ideal, bi-ideal, interior ideal and (1,2)-ideal in terms of intuitionistic hesitant fuzzy subsemigroups of semigroups. Some relations between intuitionistic hesitant fuzzy subsemigroups and subsemigroups were obtained. In the future, we will characterize filter subsemigroups in terms of intuitionistic hesitant fuzzy filter subsemigroups of semigroups.

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