วารสารวิทยาศาสตร์บูรพา ปีที่ 26 (ฉบับที่ 1) มกราคม – เมษายน พ.ศ. 2564
BURAPHA SCIENCE JOURNAL Volume 26 (No.1) January – April 2021 บทความวิจัย

ลำดับเมทริกซ์ในพจน์ของพหุนามเกาส์เซียนเพลล์ พหุนามเกาส์เซียนโมดิฟายด์เพลล์ จำนวนเกาส์เซียนเพลล์ จำนวนเกาส์เซียนเพลล์-ลูคัส จำนวนเกาส์เซียนโมดิฟายด์เพลล์ พหุนามเพลล์ พหุนามเพลล์-ลูคัส และพหุนามโมดิฟายด์เพลล์

Matrix Sequences in Terms of Gaussian Pell Polynomial, Gaussian Modified Pell Polynomial, Gaussian Pell Number, Gaussian Pell-Lucas Number, Gaussian Modified Pell Number, Pell Polynomial, Pell-Lucas Polynomial and Modified Pell Polynomial

มงคล ทาทอง*

Mongkol Tatong*

ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์ คณะวิทยาศาสตร์และเทคโนโลยี มหาวิทยาลัยเทคโนโลยีราชมงคลธัญบุรี

Department of Mathematics and Computer Science, Faculty of Science and Technology,

Rajamangala University of Technology Thanyaburi

Received : 29 February 2020

Revised : 8 May 2020

Accepted : 9 July 2020

บทคัดย่อ

ในบทความนี้เราได้ศึกษาลำดับเมทริกซ์พหุนามเกาส์เซียนเพลล์ ลำดับเมทริกซ์พหุนามเกาส์เซียนโมดิฟายด์เพลล์ ลำดับเมทริกซ์เกาส์เซียนเพลล์ ลำดับเมทริกซ์เกาส์เซียนเพลล์-ลูคัส ลำดับเมทริกซ์เกาส์เซียนโมดิฟายด์เพลล์ ลำดับเมทริกซ์พหุนามเพลล์ ลำดับเมทริกซ์พหุนามเพลล์-ลูคัส และลำดับเมทริกซ์พหุนามโมดิฟายด์เพลล์ พร้อมทั้งพิสูจน์เอกลักษณ์บางอย่างของความสัมพันธ์ระหว่างลำดับเมทริกซ์และเอกลักษณ์บางอย่างของผลบวก

คำสำคัญ : ความสัมพันธ์เวียนเกิด ; ลำดับเมทริกซ์ ; สูตรไบเนต ; พจน์ที่ n

Abstract

In this paper, we study Gaussian Pell polynomial, Gaussian modified Pell polynomial, Gaussian Pell, Gaussian Pell-Lucas, Gaussian modified Pell, Pell polynomial, Pell-Lucas polynomial, and modified Pell polynomial matrix sequences. Furthermore, we prove some identities of the relation between matrix sequences and summations.

Keywords : recurrence relations ; matrix sequences ; Binet's formulas ; n^{th} terms

*Corresponding author. E-mail : mongkol_t@rmutt.ac.th

Introduction

The Fibonacci numbers $\boldsymbol{F_n}$, Lucas numbers $\boldsymbol{L_n}$, Pell numbers $\boldsymbol{P_n}$, and Pell-Lucus numbers $\boldsymbol{Q_n}$ are examples of the famous number generated by recurrence relation. Their Binet's formulas are $F_n = \frac{r_1^{\,n} - r_2^{\,n}}{r_1 - r_2}$, $L_n = r_1^{\,n} + r_2^{\,n}$, $P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$, and $Q_n = \alpha^n + \beta^n$, where n is an integer, $r_1 = \frac{1}{2}\left(1+\sqrt{5}\right)$, $r_2 = \frac{1}{2}\left(1-\sqrt{5}\right)$ are roots of $t^2 - t - 1 = 0$ and $\alpha = 1+\sqrt{2}$, $\beta = 1-\sqrt{2}$ are roots of $t^2 - 2t - 1 = 0$. So $r_1 - r_2 = \sqrt{5}$, $r_1 r_2 = -1$, $\alpha \neq \beta$, $\alpha + \beta = 2$, $\alpha - \beta = 2\sqrt{2}$, and $\alpha\beta = -1$. (Horadam, A.F., 1961), (Daykin, D.E. & Dresel, L.A.G., 1967), (Horadam, A.F., 1984).

In 1985, Alwyn F. Horadam and Brother J.M. Mahon studied properties of the sequences of Pell polynomials $\boldsymbol{P_n(x)}$ and Pell-Lucas polynomials $\boldsymbol{Q_n(x)}$. For a natural number n, these sequences are defined by the recurrence relations

$$P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x), \tag{1}$$

and

$$Q_n(x) = 2xQ_{n-1}(x) + Q_{n-2}(x), \tag{2}$$

with initial conditions $P_0(x) = 0$, $P_1(x) = 1$, $Q_0(x) = 2$, and $Q_1(x) = 2x$.

The definitions of negative subscript are extended by

$$P_{-n}(x) = (-1)^{n+1} P_n(x), \text{ for } n \geq 1, \tag{3}$$

and

$$Q_{-n}(x) = (-1)^{n} Q_n(x), \text{ for } n \geq 1. \tag{4}$$

So, Binet's formulas can be derived as follows

$$P_n(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)}, \tag{5}$$

and

$$Q_n(x) = \alpha^n(x) + \beta^n(x), \tag{6}$$

where $\alpha(x) = x + \sqrt{x^2+1}$ and $\beta(x) = x - \sqrt{x^2+1}$ are the roots of $t^2 - 2xt - 1 = 0$. Then $\alpha(x) \neq \beta(x)$, $\alpha(x) + \beta(x) = 2x$, $\alpha(x) - \beta(x) = 2\sqrt{x^2+1}$ and $\alpha(x)\beta(x) = -1$. By (5) and (6), we have the following elementary identity:

$$P_{n+1}(x) + P_{n-1}(x) = Q_n(x). \tag{7}$$

The particular cases of the polynomials are $P_n(1) = P_n$, $Q_n(1) = Q_n$, $P_n\left(\frac{1}{2}\right) = F_n$, and $Q_n\left(\frac{1}{2}\right) = L_n$.

In 2012, Hasan Huseyin Gulec and Necati Taskara studied the (s,t)-Pell matrix sequence $\{\boldsymbol{P_n(s,t)}\}_{n\in\mathbb{N}}$ and (s,t)-Pell-Lucas matrix sequence $\{\boldsymbol{Q_n(s,t)}\}_{n\in\mathbb{N}}$ consisting of elements of the (s,t)-Pell numbers and (s,t)-Pell-Lucas numbers defined by

$$P_n(s,t) = 2sP_{n-1}(s,t) + tP_{n-2}(s,t), \text{ for } n \geq 2, \tag{8}$$

and

$$Q_n(s,t) = 2sQ_{n-1}(s,t) + tQ_{n-2}(s,t), \text{ for } n \geq 2, \tag{9}$$

with initial conditions $P_0(s,t)=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, $P_1(s,t)=\begin{pmatrix}2s & 1\\ t & 0\end{pmatrix}$, $Q_0(s,t)=\begin{pmatrix}2s & 2\\ 2t & -2s\end{pmatrix}$, and $Q_1(s,t)=\begin{pmatrix}4s^2+2t & 2s\\ 2st & 2t\end{pmatrix}$, where $s^2+t>0$, $s>0$, $t\neq 0$, and s, t are real numbers.

In 2016, Serpil Halici and Sinan Öz introduced the complex Pell and complex Pell – Lucas sequences, namely Gaussian Pell sequence $\{\boldsymbol{GP_n}\}_{n\in\mathbb{N}}$ and Gaussian Pell-Lucas sequence $\{\boldsymbol{GQ_n}\}_{n\in\mathbb{N}}$, which are defined by recurrence relations

$$GP_n=2GP_{n-1}+GP_{n-2},\text{ for } n\geq 2, \tag{10}$$

and
$$GQ_n=2GQ_{n-1}+GQ_{n-2},\text{ for } n\geq 2, \tag{11}$$

with initial conditions $GP_0=i$, $GP_1=1$, $GQ_0=2-2i$ and $GQ_1=2+2i$. Their well-known Binet's formulas are

$$GP_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}+i\frac{\alpha\beta^n-\beta\alpha^n}{\alpha-\beta}, \tag{12}$$

and
$$GQ_n=\alpha^n+\beta^n-i\alpha\beta^n-i\beta\alpha^n. \tag{13}$$

Also, Gaussian Pell and Gaussian Pell-Lucas are related to Pell and Pell-Lucas. Some identities of the sequences are

$$GP_n=P_n+iP_{n-1},\text{ for } n\geq 1, \tag{14}$$

and
$$GQ_n=Q_n+iQ_{n-1},\text{ for } n\geq 1, \tag{15}$$

with initial conditions $P_0=0$, $P_1=1$, $Q_0=2$ and $Q_1=2$.

In 2018, Tulay Yagmar and Nusret karaaslan defined Gaussian modified Pell numbers $\boldsymbol{Gq_n}$ and Gaussian modified Pell polynomials $\boldsymbol{Gq_n(x)}$ by

$$Gq_n=2Gq_{n-1}+Gq_{n-2},\text{ for } n\geq 2, \tag{16}$$

and
$$Gq_n(x)=2xGq_{n-1}(x)+Gq_{n-2}(x),\text{ for } n\geq 2, \tag{17}$$

with initial conditions $Gq_0=1-i$, $Gq_1=1+i$, $Gq_0(x)=1-xi$ and $Gq_1(x)=x+i$. Then, their Binet's formulas are

$$Gq_n=\frac{\alpha^n+\beta^n}{2}-i\frac{\alpha\beta^n+\beta\alpha^n}{2}, \tag{18}$$

and
$$Gq_n(x)=\frac{\alpha^n(x)+\beta^n(x)}{2}-i\frac{\alpha(x)\beta^n(x)+\beta(x)\alpha^n(x)}{2}. \tag{19}$$

In 2018, Serpil Halici and Sinan Oz introduced Gaussian Pell polynomials $\boldsymbol{GP_n(x)}$, which is defined recurrently by

$$GP_{n+1}(x)=2xGP_n(x)+GP_{n-1}(x),\text{ for } n\geq 1, \tag{20}$$

with initial conditions $GP_0(x)=i$ and $GP_1(x)=1$. Their well-known Binet's formula is

$$GP_n(x)=\frac{\alpha^n(x)-\beta^n(x)}{\alpha(x)-\beta(x)}+i\frac{\alpha(x)\beta^n(x)-\beta(x)\alpha^n(x)}{\alpha(x)-\beta(x)}. \tag{21}$$

That authors observed that relation between Gaussian Pell polynomial and Pell polynomial is

$$GP_n(x) = P_n(x) + iP_{n-1}(x), \text{ for } n \geq 1. \tag{22}$$

In 2019, Nusret Karaaslan studied modified Pell polynomials $\boldsymbol{q_n(x)}$ defined by

$$q_n(x) = 2xq_{n-1}(x) + q_{n-2}(x), \text{ for } n \geq 2, \tag{23}$$

with initial conditions $q_0(x) = 1$ and $q_1(x) = x$. Then, the Binet's formula is

$$q_n(x) = \frac{\alpha^n(x) + \beta^n(x)}{2}. \tag{24}$$

In particular, if $x = 1$, then $q_n(1)$ is the modified Pell number q_n.

Methods

In this section, firstly, we find the first few terms of the recurrence relations GP_n, GQ_n, Gq_n, $Gq_n(x)$, $GP_n(x)$, and $q_n(x)$, which the extension of negative subscripts is created by rewriting as $GP_{n-2} = GP_n - 2GP_{n-1}$, $GQ_{n-2} = GQ_n - 2GQ_{n-1}$, $Gq_{n-2} = Gq_n - 2Gq_{n-1}$, $Gq_{n-2}(x) = Gq_n(x) - 2xGq_{n-1}(x)$, $GP_{n-2}(x) = GP_n(x) - 2xGP_{n-1}(x)$, and $q_{n-2}(x) = q_n(x) - 2xq_{n-1}(x)$ as below.

Table 1 The first few terms of GP_n, GQ_n, Gq_n, $Gq_n(x)$, $GP_n(x)$, and $q_n(x)$ for $-2 \leq n \leq 1$.

n:	-2	-1	0	1
GP_n:	$-2+5i$	$1-2i$	i	1
GQ_n:	$6-14i$	$-2+6i$	$2-2i$	$2+2i$
Gq_n:	$3-7i$	$-1+3i$	$1-i$	$1+i$
$Gq_n(x)$:	$(2x^2+1)-(4x^3+3x)i$	$-x+(2x^2+1)i$	$1-xi$	$x+i$
$GP_n(x)$:	$-2x+(4x^2+1)i$	$1-2xi$	i	1
$q_n(x)$:	$2x^2+1$	$-x$	1	x

After that, we define the recurrence relation of a 2x2 matrix for all integer $n \geq -1$ in which the component of each matrix consists of numbers and polynomials of these sequences, and the index starts at -1.

Definition 1 Let $n \in \mathbb{N}$, x is a scalar-value polynomial, $x > 0$, and $x^2 + 1 > 0$. Then the Gaussian Pell polynomial matrix sequence $\{\boldsymbol{MGP_n(x)}\}_{n\in\mathbb{N}}$ and Gaussian modified Pell polynomial matrix sequence $\{\boldsymbol{MGq_n(x)}\}_{n\in\mathbb{N}}$ are defined by

$$MGP_n(x) = 2xMGP_{n-1}(x) + MGP_{n-2}(x), \tag{25}$$

and

$$MGq_n(x) = 2xMGq_{n-1}(x) + MGq_{n-2}(x), \tag{26}$$

respectively, with initial conditions $MGP_0(x)=\begin{pmatrix}1 & i\\ i & 1-2xi\end{pmatrix}$, $MGP_1(x)=\begin{pmatrix}2x+i & 1\\ 1 & i\end{pmatrix}$, $MGq_0(x)=\begin{pmatrix}x+i & 1-xi\\ 1-xi & -x+(2x^2+1)i\end{pmatrix}$, and $MGq_1(x)=\begin{pmatrix}2x^2+1+xi & x+i\\ x+i & 1-xi\end{pmatrix}$.

Definition 2 Let $n\in\mathbb{N}$. Then the Gaussian Pell matrix sequence $\{\boldsymbol{MGP_n}\}_{n\in\mathbb{N}}$, Gaussian Pell-Lucas matrix sequence $\{\boldsymbol{MGQ_n}\}_{n\in\mathbb{N}}$, and Gaussian modified Pell matrix sequence $\{\boldsymbol{MGq_n}\}_{n\in\mathbb{N}}$ are defined by

$$MGP_n=2MGP_{n-1}+MGP_{n-2}, \tag{27}$$

and

$$MGQ_n=2MGQ_{n-1}+MGQ_{n-2}, \tag{28}$$

and

$$MGq_n=2MGq_{n-1}+MGq_{n-2}, \tag{29}$$

respectively, with initial conditions $MGP_0=\begin{pmatrix}1 & i\\ i & 1-2i\end{pmatrix}$, $MGP_1=\begin{pmatrix}2+i & 1\\ 1 & i\end{pmatrix}$, $MGQ_0=\begin{pmatrix}2+2i & 2-2i\\ 2-2i & -2+6i\end{pmatrix}$, $MGQ_1=\begin{pmatrix}6+2i & 2+2i\\ 2+2i & 2-2i\end{pmatrix}$, $MGq_0=\begin{pmatrix}1+i & 1-i\\ 1-i & -1+3i\end{pmatrix}$, and $MGq_1=\begin{pmatrix}3+i & 1+i\\ 1+i & 1-i\end{pmatrix}$.

Definition 3 Let $n\in\mathbb{N}$, x is a scalar-value polynomial, $x>0$, and $x^2+1>0$. Then the Pell polynomial matrix sequence $\{\boldsymbol{MP_n(x)}\}_{n\in\mathbb{N}}$, Pell-Lucas polynomial matrix sequence $\{\boldsymbol{MQ_n(x)}\}_{n\in\mathbb{N}}$, and modified Pell polynomial matrix sequence $\{\boldsymbol{Mq_n(x)}\}_{n\in\mathbb{N}}$ are defined by

$$MP_n(x)=2xMP_{n-1}(x)+MP_{n-2}(x), \tag{30}$$

and

$$MQ_n(x)=2xMQ_{n-1}(x)+MQ_{n-2}(x), \tag{31}$$

and

$$Mq_n(x)=2xMq_{n-1}(x)+Mq_{n-2}(x), \tag{32}$$

respectively, with initial conditions $MP_0(x)=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, $MP_1(x)=\begin{pmatrix}2x & 1\\ 1 & 0\end{pmatrix}$, $MQ_0(x)=\begin{pmatrix}2x & 2\\ 2 & -2x\end{pmatrix}$, $MQ_1(x)=\begin{pmatrix}4x^2+2 & 2x\\ 2x & 2\end{pmatrix}$, $Mq_0(x)=\begin{pmatrix}x & 1\\ 1 & -x\end{pmatrix}$, and $Mq_1(x)=\begin{pmatrix}2x^2+1 & x\\ x & 1\end{pmatrix}$.

Note that, for all integer $n<0$, we find negative subscripts of matrix sequences in which the extension of definition is obtained by rewriting

$$MGP_{n-2}(x)=MGP_n(x)-2xMGP_{n-1}(x), \tag{33}$$

$$MGq_{n-2}(x)=MGq_n(x)-2xMGq_{n-1}(x), \tag{34}$$

$$MGP_{n-2}=MGP_n-2MGP_{n-1}, \tag{35}$$

$$MGQ_{n-2}=MGQ_n-2MGQ_{n-1}, \tag{36}$$

$$MGq_{n-2} = MGq_n - 2MGq_{n-1}, \tag{37}$$

$$MP_{n-2}(x) = MP_n(x) - 2xMP_{n-1}(x), \tag{38}$$

$$MQ_{n-2}(x) = MQ_n(x) - 2xMQ_{n-1}(x), \tag{39}$$

and

$$Mq_{n-2}(x) = Mq_n(x) - 2xMq_{n-1}(x). \tag{40}$$

Results

In this section, the first step, we find the n[th] general terms of the matrix sequences, which correspond to the following theorem and corollary.

Theorem 4 For $n \in \mathbb{N}$. Then the n[th] terms of $\{MGP_n(x)\}_{n\in\mathbb{N}}$ and $\{MGq_n(x)\}_{n\in\mathbb{N}}$ are given by

$$MGP_n(x) = \begin{pmatrix} GP_{n+1}(x) & GP_n(x) \\ GP_n(x) & GP_{n-1}(x) \end{pmatrix}, \tag{41}$$

and

$$MGq_n(x) = \begin{pmatrix} Gq_{n+1}(x) & Gq_n(x) \\ Gq_n(x) & Gq_{n-1}(x) \end{pmatrix}. \tag{42}$$

Proof. We will show that $MGP_n(x) = \begin{pmatrix} GP_{n+1}(x) & GP_n(x) \\ GP_n(x) & GP_{n-1}(x) \end{pmatrix}$ for $n \in \mathbb{N}$.

Since, $MGP_0(x) = \begin{pmatrix} 1 & i \\ i & 1-2xi \end{pmatrix}$, it follows that (41) is true.

Since, $MGP_1(x) = \begin{pmatrix} 2x+i & 1 \\ 1 & i \end{pmatrix}$, it follows that (41) is true.

By iterating this procedure and considering induction steps, let us assume that the equality in (41) holds for all $n \le k \in \mathbb{N}$. To finish the proof.

Next, we have to show that (41) also holds for $n = k+1$ by considering (20) and (25).

Then

$$\begin{aligned} MGP_{k+1}(x) &= 2xMGP_k(x) + MGP_{k-1}(x) \\ &= 2x\begin{pmatrix} GP_{k+1}(x) & GP_k(x) \\ GP_k(x) & GP_{k-1}(x) \end{pmatrix} + \begin{pmatrix} GP_k(x) & GP_{k-1}(x) \\ GP_{k-1}(x) & GP_{k-2}(x) \end{pmatrix} \\ &= \begin{pmatrix} 2xGP_{k+1}(x) + GP_k(x) & 2xGP_k(x) + GP_{k-1}(x) \\ 2xGP_k(x) + GP_{k-1}(x) & 2xGP_{k-1}(x) + GP_{k-2}(x) \end{pmatrix} \\ &= \begin{pmatrix} GP_{k+2}(x) & GP_{k+1}(x) \\ GP_{k+1}(x) & GP_k(x) \end{pmatrix}. \end{aligned}$$

Thus, $n = k+1$ is true.

The similar proof of (41) is used to prove (42).

Therefore, the proof is complete.

Corollary 5 For $n \in \mathbb{N}$. Then the n[th] terms of $\{MGP_n\}_{n\in\mathbb{N}}$, $\{MGQ_n\}_{n\in\mathbb{N}}$, and $\{MGq_n\}_{n\in\mathbb{N}}$ are given by

$$MGP_n = \begin{pmatrix} GP_{n+1} & GP_n \\ GP_n & GP_{n-1} \end{pmatrix}, \tag{43}$$

and

$$MGQ_n = \begin{pmatrix} GQ_{n+1} & GQ_n \\ GQ_n & GQ_{n-1} \end{pmatrix}, \tag{44}$$

and

$$MGq_n = \begin{pmatrix} Gq_{n+1} & Gq_n \\ Gq_n & Gq_{n-1} \end{pmatrix}. \tag{45}$$

Proof. Take $x=1$ in (41) and (42), we have (43) and (45).

The similar proof of Theorem 4 is used for (44).

Corollary 6 For $n \in \mathbb{N}$. Then the n[th] terms of $\{MP_n(x)\}_{n\in\mathbb{N}}$, $\{MQ_n(x)\}_{n\in\mathbb{N}}$, and $\{Mq_n(x)\}_{n\in\mathbb{N}}$ are given by

$$MP_n(x) = \begin{pmatrix} P_{n+1}(x) & P_n(x) \\ P_n(x) & P_{n-1}(x) \end{pmatrix}, \tag{46}$$

and

$$MQ_n(x) = \begin{pmatrix} Q_{n+1}(x) & Q_n(x) \\ Q_n(x) & Q_{n-1}(x) \end{pmatrix}, \tag{47}$$

and

$$Mq_n(x) = \begin{pmatrix} q_{n+1}(x) & q_n(x) \\ q_n(x) & q_{n-1}(x) \end{pmatrix}. \tag{48}$$

Proof. The similar proof of Theorem 4 is used for (46), (47), and (48).

Next, we find Binet's formulas of the matrix sequences that lead to some identities. These formulas correspond to the following theorem and corollary.

Theorem 7 Let $n \in \mathbb{N}$. Then, the Binet's formula for $\{MGP_n(x)\}_{n\in\mathbb{N}}$ and $\{MGq_n(x)\}_{n\in\mathbb{N}}$ are given by

$$MGP_n(x) = \frac{\alpha^n(x)}{\alpha(x)-\beta(x)}\big(MGP_1(x)-\beta(x)MGP_0(x)\big) - \frac{\beta^n(x)}{\alpha(x)-\beta(x)}\big(MGP_1(x)-\alpha(x)MGP_0(x)\big), \tag{49}$$

and

$$MGq_n(x) = \frac{\alpha^n(x)}{\alpha(x)-\beta(x)}\big(MGq_1(x)-\beta(x)MGq_0(x)\big) - \frac{\beta^n(x)}{\alpha(x)-\beta(x)}\big(MGq_1(x)-\alpha(x)MGq_0(x)\big). \tag{50}$$

Proof. Let c_1, c_2 be the 2x2 matrices and $\alpha(x) = x+\sqrt{x^2+1}$, $\beta(x) = x-\sqrt{x^2+1}$ be the roots of $t^2-2xt-1=0$. Then the general term of (25) is

$$MGP_n(x) = c_1\alpha^n(x) + c_2\beta^n(x). \tag{51}$$

Take $n=0$ and $n=1$ in (51), we get

$$MGP_0(x)=c_1+c_2, \tag{52}$$

and

$$MGP_1(x)=c_1\alpha(x)+c_2\beta(x). \tag{53}$$

By using (52), (53) and scalar multiplication to find c_1 and c_2, we obtain

$$c_1=\frac{1}{\alpha(x)-\beta(x)}\left(MGP_1(x)-\beta(x)MGP_0(x)\right), \tag{54}$$

and

$$c_2=-\frac{1}{\alpha(x)-\beta(x)}\left(MGP_1(x)-\alpha(x)MGP_0(x)\right). \tag{55}$$

By using (54), (55) in (51), we get

$$MGP_n(x)=\frac{\alpha^n(x)}{\alpha(x)-\beta(x)}\left(MGP_1(x)-\beta(x)MGP_0(x)\right)-\frac{\beta^n(x)}{\alpha(x)-\beta(x)}\left(MGP_1(x)-\alpha(x)MGP_0(x)\right).$$

The similar proof of (49) is used to prove (50).

Therefore, the proof is complete.

Corollary 8 Let $n\in\mathbb{N}$. Then, the Binet's formula for $\{MGP_n\}_{n\in\mathbb{N}}$, $\{MGQ_n\}_{n\in\mathbb{N}}$, and $\{MGq_n\}_{n\in\mathbb{N}}$ are given by

$$MGP_n=\frac{\alpha^n}{\alpha-\beta}\left(MGP_1-\beta MGP_0\right)-\frac{\beta^n}{\alpha-\beta}\left(MGP_1-\alpha MGP_0\right), \tag{56}$$

and

$$MGQ_n=\frac{\alpha^n}{\alpha-\beta}\left(MGQ_1-\beta MGQ_0\right)-\frac{\beta^n}{\alpha-\beta}\left(MGQ_1-\alpha MGQ_0\right), \tag{57}$$

and

$$MGq_n=\frac{\alpha^n}{\alpha-\beta}\left(MGq_1-\beta MGq_0\right)-\frac{\beta^n}{\alpha-\beta}\left(MGq_1-\alpha MGq_0\right). \tag{58}$$

Proof. Take $x=1$ in (49) and (50), we have (56) and (58).

The similar proof of Theorem 7 is used to (57).

Corollary 9 Let $n\in\mathbb{N}$. Then, the Binet's formula for $\{MP_n(x)\}_{n\in\mathbb{N}}$, $\{MQ_n(x)\}_{n\in\mathbb{N}}$, and $\{Mq_n(x)\}_{n\in\mathbb{N}}$ are given by

$$MP_n(x)=\frac{\alpha^n(x)}{\alpha(x)-\beta(x)}\left(MP_1(x)-\beta(x)MP_0(x)\right)-\frac{\beta^n(x)}{\alpha(x)-\beta(x)}\left(MP_1(x)-\alpha(x)MP_0(x)\right), \tag{59}$$

and

$$MQ_n(x)=\frac{\alpha^n(x)}{\alpha(x)-\beta(x)}\left(MQ_1(x)-\beta(x)MQ_0(x)\right)-\frac{\beta^n(x)}{\alpha(x)-\beta(x)}\left(MQ_1(x)-\alpha(x)MQ_0(x)\right), \tag{60}$$

and

$$Mq_n(x)=\frac{\alpha^n(x)}{\alpha(x)-\beta(x)}\left(Mq_1(x)-\beta(x)Mq_0(x)\right)-\frac{\beta^n(x)}{\alpha(x)-\beta(x)}\left(Mq_1(x)-\alpha(x)Mq_0(x)\right). \tag{61}$$

Proof. The similar proof of Theorem 7 is used to prove (59), (60), and (61).

Next, we find the n[th] power of $MP_1(x)$ and MP_1, for any integer $n \geq 0$, which are symmetry. They correspond to the following lemma.

Lemma 10 For $n \in \mathbb{N}$, the n[th] power of $MP_1(x)$ and MP_1 are given by

(i) $\left(MP_1(x)\right)^n = MP_n(x)$,

(ii) $\left(MP_1\right)^n = MP_n$.

Proof. Since, $MP_0(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, it follows that (i) is true.

Since, $MP_1(x) = \begin{pmatrix} 2x & 1 \\ 1 & 0 \end{pmatrix}$, it follows that (i) is true.

We assume the result is true for a positive integer $n = k$ then

$$MP_k(x) = \begin{pmatrix} P_{k+1}(x) & P_k(x) \\ P_k(x) & P_{k-1}(x) \end{pmatrix} = \left(MP_1(x)\right)^k.$$

We consider a positive integer $n = k+1$.

Then
$$\begin{aligned}
\left(MP_1(x)\right)^{k+1} &= \left(MP_1(x)\right)^k MP_1(x) \\
&= \begin{pmatrix} P_{k+1}(x) & P_k(x) \\ P_k(x) & P_{k-1}(x) \end{pmatrix} \begin{pmatrix} 2x & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 2xP_{k+1}(x) + P_k(x) & P_{k+1}(x) \\ 2xP_k(x) + P_{k-1}(x) & P_k(x) \end{pmatrix} \\
&= \begin{pmatrix} P_{k+2}(x) & P_{k+1}(x) \\ P_{k+1}(x) & P_k(x) \end{pmatrix} \\
&= MP_{k+1}(x).
\end{aligned}$$

Thus, the statement is true when $n = k+1$.

Take $x = 1$ in (i), we have (ii).

Therefore, the proof is complete.

After that, we find some identities of the relations between the studied sequences of numbers and polynomials, which corresponds to the following lemma.

Lemma 11 For $m, n \in \mathbb{N}$, the following results hold.

(i) $GP_{m+1}(x)P_n(x) + GP_m(x)P_{n-1}(x) = GP_{m+n}(x)$,

(ii) $Gq_{m+1}(x)P_n(x) + Gq_m(x)P_{n-1}(x) = Gq_{m+n}(x)$,

(iii) $GP_{m+1}P_n + GP_mP_{n-1} = GP_{m+n}$,

(iv) $GQ_{m+1}P_n + GQ_mP_{n-1} = GQ_{m+n}$,

(v) $Gq_{m+1}P_n + Gq_mP_{n-1} = Gq_{m+n}$,

(vi) $P_{m+1}(x)P_n(x)+P_m(x)P_{n-1}(x)=P_{m+n}(x)$,

(vii) $Q_{m+1}(x)P_n(x)+Q_m(x)P_{n-1}(x)=Q_{m+n}(x)$,

(viii) $q_{m+1}(x)P_n(x)+q_m(x)P_{n-1}(x)=q_{m+n}(x)$,

(ix) $GP_{n+1}(x)+GP_{n-1}(x)=2Gq_n(x)$,

(x) $GP_{n+1}+GP_{n-1}=2Gq_n$,

(xi) $P_{n+1}(x)Q_{n+1}(x)+P_n(x)Q_n(x)=P_{2n+2}(x)+P_{2n}(x)=Q_{2n+1}(x)$,

(xii) $P_n(x)Q_{n+1}(x)+P_{n-1}(x)Q_n(x)=P_{2n+1}(x)+P_{2n-1}(x)=Q_{2n}(x)=P_{n+1}(x)Q_n(x)+P_n(x)Q_{n-1}(x)$,

(xiii) $P_n(x)Q_n(x)+P_{n-1}(x)Q_{n-1}(x)=P_{2n}(x)+P_{2n-2}(x)=Q_{2n-1}(x)$.

Proof. Since (5), (21) and $\alpha(x)=x+\sqrt{x^2+1}$, $\beta(x)=x-\sqrt{x^2+1}$ be the roots of $t^2-2xt-1=0$, we have

$$
\begin{aligned}
&GP_{m+1}(x)P_n(x)+GP_m(x)P_{n-1}(x)\\
&=\left(\frac{\alpha^{m+1}(x)-\beta^{m+1}(x)}{\alpha(x)-\beta(x)}+i\frac{\alpha(x)\beta^{m+1}(x)-\beta(x)\alpha^{m+1}(x)}{\alpha(x)-\beta(x)}\right)\cdot\left(\frac{\alpha^{n}(x)-\beta^{n}(x)}{\alpha(x)-\beta(x)}\right)\\
&\qquad+\left(\frac{\alpha^{m}(x)-\beta^{m}(x)}{\alpha(x)-\beta(x)}+i\frac{\alpha(x)\beta^{m}(x)-\beta(x)\alpha^{m}(x)}{\alpha(x)-\beta(x)}\right)\cdot\left(\frac{\alpha^{n-1}(x)-\beta^{n-1}(x)}{\alpha(x)-\beta(x)}\right)\\
&=\frac{1}{(\alpha(x)-\beta(x))^2}\big(\alpha^{m+n+1}(x)-\alpha^{m+1}(x)\beta^{n}(x)-\alpha^{n}(x)\beta^{m+1}(x)+\beta^{m+n+1}(x)\\
&\qquad+\alpha^{m+n-1}(x)-\alpha^{m}(x)\beta^{n-1}(x)-\alpha^{n-1}(x)\beta^{m}(x)+\beta^{m+n-1}(x)\\
&\qquad+i\alpha^{n+1}(x)\beta^{m+1}(x)-i\alpha(x)\beta^{m+n+1}(x)-i\beta(x)\alpha^{m+n+1}(x)+i\alpha^{m+1}(x)\beta^{n+1}(x)\\
&\qquad+i\alpha^{n}(x)\beta^{m}(x)-i\alpha(x)\beta^{m+n-1}(x)-i\beta(x)\alpha^{m+n-1}(x)+i\alpha^{m}(x)\beta^{n}(x)\big)\\
&=\frac{1}{\alpha(x)-\beta(x)}\Bigg(\alpha^{m+n}(x)\frac{\alpha(x)+\alpha^{-1}(x)}{\alpha(x)-\beta(x)}-\alpha^{m}(x)\beta^{n}(x)\frac{\alpha(x)+\beta^{-1}(x)}{\alpha(x)-\beta(x)}-\alpha^{n}(x)\beta^{m}(x)\frac{\beta(x)+\alpha^{-1}(x)}{\alpha(x)-\beta(x)}\\
&\qquad+\beta^{m+n}(x)\frac{\beta(x)+\beta^{-1}(x)}{\alpha(x)-\beta(x)}+i\alpha^{n}(x)\beta^{m}(x)\frac{\alpha(x)\beta(x)+1}{\alpha(x)-\beta(x)}-i\alpha(x)\beta^{m+n}(x)\frac{\beta(x)+\beta^{-1}(x)}{\alpha(x)-\beta(x)}\\
&\qquad-i\beta(x)\alpha^{m+n}(x)\frac{\alpha(x)+\alpha^{-1}(x)}{\alpha(x)-\beta(x)}+i\alpha^{m}(x)\beta^{n}(x)\frac{\alpha(x)\beta(x)+1}{\alpha(x)-\beta(x)}\Bigg)\\
&=\frac{1}{\alpha(x)-\beta(x)}\big(\alpha^{m+n}(x)-\beta^{m+n}(x)+i\alpha(x)\beta^{m+n}(x)-i\beta(x)\alpha^{m+n}(x)\big)\\
&=\frac{\alpha^{m+n}(x)-\beta^{m+n}(x)}{\alpha(x)-\beta(x)}+i\frac{\alpha(x)\beta^{m+n}(x)-\beta(x)\alpha^{m+n}(x)}{\alpha(x)-\beta(x)}\\
&=GP_{m+n}(x).
\end{aligned}
$$

Thus, $GP_{m+1}(x)P_n(x)+GP_m(x)P_{n-1}(x)=GP_{m+n}(x)$.

The similar proof of (i) is used for (ii), (iii), (iv), (v), (vi), (vii), and (viii).

Next, we will show that $GP_{n+1}(x)+GP_{n-1}(x)=2Gq_n(x)$.

By (21), we have

$$\begin{aligned}
&GP_{n+1}(x)+GP_{n-1}(x)\\
&\quad=\frac{\alpha^{n+1}(x)-\beta^{n+1}(x)}{\alpha(x)-\beta(x)}+i\frac{\alpha(x)\beta^{n+1}(x)-\beta(x)\alpha^{n+1}(x)}{\alpha(x)-\beta(x)}+\frac{\alpha^{n-1}(x)-\beta^{n-1}(x)}{\alpha(x)-\beta(x)}+i\frac{\alpha(x)\beta^{n-1}(x)-\beta(x)\alpha^{n-1}(x)}{\alpha(x)-\beta(x)}\\
&\quad=\alpha^{n}(x)\frac{\alpha(x)+\alpha^{-1}(x)}{\alpha(x)-\beta(x)}-\beta^{n}(x)\frac{\beta(x)+\beta^{-1}(x)}{\alpha(x)-\beta(x)}\\
&\qquad\qquad+i\alpha(x)\beta^{n}(x)\frac{\beta(x)+\beta^{-1}(x)}{\alpha(x)-\beta(x)}-i\beta(x)\alpha^{n}(x)\frac{\alpha(x)+\alpha^{-1}(x)}{\alpha(x)-\beta(x)}\\
&\quad=\alpha^{n}(x)+\beta^{n}(x)-i\alpha(x)\beta^{n}(x)-i\beta(x)\alpha^{n}(x)\\
&\quad=2Gq_n(x).
\end{aligned}$$

Thus, $GP_{n+1}(x)+GP_{n-1}(x)=2Gq_n(x)$.

Take $x=1$ in (ix), we have (x).

Next, we will show that $P_{n+1}(x)Q_{n+1}(x)+P_n(x)Q_n(x)=P_{2n+2}(x)+P_{2n}(x)=Q_{2n+1}(x)$.

By (5) and (6), we have

$$\begin{aligned}
&P_{n+1}(x)Q_{n+1}(x)+P_n(x)Q_n(x)\\
&\quad=\left(\frac{\alpha^{n+1}(x)-\beta^{n+1}(x)}{\alpha(x)-\beta(x)}\right)\cdot\left(\alpha^{n+1}(x)+\beta^{n+1}(x)\right)+\left(\frac{\alpha^{n}(x)-\beta^{n}(x)}{\alpha(x)-\beta(x)}\right)\cdot\left(\alpha^{n}(x)+\beta^{n}(x)\right)\\
&\quad=\frac{\alpha^{2n+2}(x)+\alpha^{n+1}(x)\beta^{n+1}(x)-\alpha^{n+1}(x)\beta^{n+1}(x)-\beta^{2n+2}(x)+\alpha^{2n}(x)+\alpha^{n}(x)\beta^{n}(x)-\alpha^{n}(x)\beta^{n}(x)-\beta^{2n}(x)}{\alpha(x)-\beta(x)}\\
&\quad=\frac{\alpha^{2n+2}(x)-\beta^{2n+2}(x)}{\alpha(x)-\beta(x)}+\frac{\alpha^{2n}(x)-\beta^{2n}(x)}{\alpha(x)-\beta(x)}\\
&\quad=P_{2n+2}(x)+P_{2n}(x).
\end{aligned} \tag{62}$$

Since (7), we obtain

$$P_{2n+2}(x)+P_{2n}(x)=Q_{2n+1}(x). \tag{63}$$

By using (62) and (63), we have

$$P_{n+1}(x)Q_{n+1}(x)+P_n(x)Q_n(x)=Q_{2n+1}(x).$$

Thus, $P_{n+1}(x)Q_{n+1}(x)+P_n(x)Q_n(x)=P_{2n+2}(x)+P_{2n}(x)=Q_{2n+1}(x)$.

The similar proof of (xi) is used for (xii) and (xiii).

Therefore, the identities (i), (ii), (iii), (iv), (v), (vi), (vii), (viii), (ix), (x), (xi), (xii), and (xiii) are immediately seen.

Also, we find the relation between these matrix sequences by applying Lemma 10 and Lemma 11. They correspond to the following theorem and corollary.

Theorem 12 For $m,n\in\mathbb{N}$, the following results hold.

(i) $MGP_m(x)\left(MP_1(x)\right)^n = MGP_{m+n}(x)$,

(ii) $MGq_m(x)\left(MP_1(x)\right)^n = MGq_{m+n}(x)$,

(iii) $MGP_{n+1}(x)+MGP_{n-1}(x)=2MGq_n(x)$.

Proof. By Lemma 10 (i), we can write

$$MGP_m(x)\left(MP_1(x)\right)^n = MGP_m(x)MP_n(x). \tag{64}$$

Since (41), (46), and matrix multiplication, then

$$\begin{aligned} MGP_m(x)MP_n(x) &= \begin{pmatrix} GP_{m+1}(x) & GP_m(x) \\ GP_m(x) & GP_{m-1}(x) \end{pmatrix}\begin{pmatrix} P_{n+1}(x) & P_n(x) \\ P_n(x) & P_{n-1}(x) \end{pmatrix} \\ &= \begin{pmatrix} GP_{m+1}(x)P_{n+1}(x)+GP_m(x)P_n(x) & GP_{m+1}(x)P_n(x)+GP_m(x)P_{n-1}(x) \\ GP_m(x)P_{n+1}(x)+GP_{m-1}(x)P_n(x) & GP_m(x)P_n(x)+GP_{m-1}(x)P_{n-1}(x) \end{pmatrix}. \end{aligned} \tag{65}$$

By using Lemma 11 (i) in (65), we have

$$\begin{aligned} MGP_m(x)MP_n(x) &= \begin{pmatrix} GP_{m+1}(x)P_{n+1}(x)+GP_m(x)P_n(x) & GP_{m+1}(x)P_n(x)+GP_m(x)P_{n-1}(x) \\ GP_m(x)P_{n+1}(x)+GP_{m-1}(x)P_n(x) & GP_m(x)P_n(x)+GP_{m-1}(x)P_{n-1}(x) \end{pmatrix} \\ &= \begin{pmatrix} GP_{m+n+1}(x) & GP_{m+n}(x) \\ GP_{m+n}(x) & GP_{m+n-1}(x) \end{pmatrix} \\ &= MGP_{m+n}(x) \end{aligned} \tag{66}$$

By using (66) in (64), we obtain

$$MGP_m(x)\left(MP_1(x)\right)^n = MGP_{m+n}(x).$$

The similar proof of (i) is used for (ii).

Next, we will show that $MGP_{n+1}(x)+MGP_{n-1}(x)=2MGq_n(x)$.

By (41) and matrix addition, we can write

$$\begin{aligned} MGP_{n+1}(x)+MGP_{n-1}(x) &= \begin{pmatrix} GP_{n+2}(x) & GP_{n+1}(x) \\ GP_{n+1}(x) & GP_n(x) \end{pmatrix}+\begin{pmatrix} GP_n(x) & GP_{n-1}(x) \\ GP_{n-1}(x) & GP_{n-2}(x) \end{pmatrix} \\ &= \begin{pmatrix} GP_{n+2}(x)+GP_n(x) & GP_{n+1}(x)+GP_{n-1}(x) \\ GP_{n+1}(x)+GP_{n-1}(x) & GP_n(x)+GP_{n-2}(x) \end{pmatrix} \end{aligned} \tag{67}$$

By using Lemma 11 (ix) in (67), we get

$$\begin{aligned} MGP_{n+1}(x)+MGP_{n-1}(x) &= \begin{pmatrix} GP_{n+2}(x)+GP_n(x) & GP_{n+1}(x)+GP_{n-1}(x) \\ GP_{n+1}(x)+GP_{n-1}(x) & GP_n(x)+GP_{n-2}(x) \end{pmatrix} \\ &= \begin{pmatrix} 2Gq_{n+1}(x) & 2Gq_n(x) \\ 2Gq_n(x) & 2Gq_{n-1}(x) \end{pmatrix} \\ &= 2\begin{pmatrix} Gq_{n+1}(x) & Gq_n(x) \\ Gq_n(x) & Gq_{n-1}(x) \end{pmatrix} \\ &= 2MGq_n(x). \end{aligned}$$

Therefore, the identities (i), (ii), and (iii) are immediately seen.

Corollary 13 For $m,n \in \mathbb{N}$, the following results hold.

(i) $MGP_m\left(MP_1\right)^n = MGP_{m+n}$,

(ii) $MGQ_m\left(MP_1\right)^n = MGQ_{m+n}$,

(iii) $MGq_m\left(MP_1\right)^n = MGq_{m+n}$,

(iv) $MGP_{n+1} + MGP_{n-1} = 2MGq_n$.

Proof. Take $x=1$ in Theorem 12 (i), (ii), and (iii), we have (i), (iii), and (iv).

The similar proof of Theorem 12 is used for (ii).

Corollary 14 For $m,n \in \mathbb{N}$, the following results hold.

(i) $MP_m(x)\left(MP_1(x)\right)^n = MP_{m+n}(x)$,

(ii) $MQ_m(x)\left(MP_1(x)\right)^n = MQ_{m+n}(x)$,

(iii) $Mq_m(x)\left(MP_1(x)\right)^n = Mq_{m+n}(x)$.

(iv) $MP_{n+1}(x) + MP_{n-1}(x) = MQ_n(x)$,

(v) $MP_n(x)MQ_n(x) = MP_{2n+1}(x) + MP_{2n-1}(x) = MQ_{2n}(x)$.

Proof. The similar proof of Theorem 12 is used for (i), (ii), (iii), and (iv).

Next, we will show that $MP_n(x)MQ_n(x) = MP_{2n+1}(x) + MP_{2n-1}(x) = MQ_{2n}(x)$.

By (46), (47) and matrix multiplication, we have

$$\begin{aligned} MP_n(x)MQ_n(x) &= \begin{pmatrix} P_{n+1}(x) & P_n(x) \\ P_n(x) & P_{n-1}(x) \end{pmatrix} \begin{pmatrix} Q_{n+1}(x) & Q_n(x) \\ Q_n(x) & Q_{n-1}(x) \end{pmatrix} \\ &= \begin{pmatrix} P_{n+1}(x)Q_{n+1}(x) + P_n(x)Q_n(x) & P_{n+1}(x)Q_n(x) + P_n(x)Q_{n-1}(x) \\ P_n(x)Q_{n+1}(x) + P_{n-1}(x)Q_n(x) & P_n(x)Q_n(x) + P_{n-1}(x)Q_{n-1}(x) \end{pmatrix}. \end{aligned} \tag{68}$$

By using Lemma 11 (xi), (xii), and (xiii) in (68), we can write

$$\begin{aligned} MP_n(x)MQ_n(x) &= \begin{pmatrix} P_{n+1}(x)Q_{n+1}(x) + P_n(x)Q_n(x) & P_{n+1}(x)Q_n(x) + P_n(x)Q_{n-1}(x) \\ P_n(x)Q_{n+1}(x) + P_{n-1}(x)Q_n(x) & P_n(x)Q_n(x) + P_{n-1}(x)Q_{n-1}(x) \end{pmatrix} \\ &= \begin{pmatrix} P_{2n+2}(x) + P_{2n}(x) & P_{2n+1}(x) + P_{2n-1}(x) \\ P_{2n+1}(x) + P_{2n-1}(x) & P_{2n}(x) + P_{2n-2}(x) \end{pmatrix} \end{aligned} \tag{69}$$

By matrix addition, we have

$$\begin{pmatrix} P_{2n+2}(x) + P_{2n}(x) & P_{2n+1}(x) + P_{2n-1}(x) \\ P_{2n+1}(x) + P_{2n-1}(x) & P_{2n}(x) + P_{2n-2}(x) \end{pmatrix} = \begin{pmatrix} P_{2n+2}(x) & P_{2n+1}(x) \\ P_{2n+1}(x) & P_{2n}(x) \end{pmatrix} + \begin{pmatrix} P_{2n}(x) & P_{2n-1}(x) \\ P_{2n-1}(x) & P_{2n-2}(x) \end{pmatrix} = MP_{2n+1}(x) + MP_{2n-1}(x) \tag{70}$$

By using (69) and (70), we get that

$$MP_n(x)MQ_n(x) = MP_{2n+1}(x) + MP_{2n-1}(x) \tag{71}$$

By (iv), we obtain

$$MP_{2n+1}(x)+MP_{2n-1}(x)=MQ_{2n}(x). \quad (72)$$

By using (71) and (72), we get

$$MP_n(x)MQ_n(x)=MQ_{2n}(x).$$

Thus, $MP_n(x)MQ_n(x)=MP_{2n+1}(x)+MP_{2n-1}(x)=MQ_{2n}(x)$.

Therefore, the identities (i), (ii), (iii), (iv), and (v) are easily seen.

Moreover, we get a particular case, which corresponds to the following corollary.

Corollary 15 For $n\in\mathbb{N}$, the following results hold.

(i) $MGP_0(x)\left(MP_1(x)\right)^n=MGP_n(x)$,

(ii) $MGP_0\left(MP_1\right)^n=MGP_n$,

(iii) $MP_0(x)\left(MP_1(x)\right)^n=MP_n(x)$.

Proof. Take $m=0$ in Theorem 12 (i), Corollary 13 (i), and Corollary 14 (i), we obtain (i), (ii), and (iii).

Therefore, the identities (i), (ii), and (iii) are easily seen.

Note that matrix $MP_0(x)=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ is an identity matrix.

Now, we find that the matrix is a symmetric matrix, which is equal to its transposition as in the following theorem and corollary.

Theorem 16 For $m,n\in\mathbb{N}$, the following results hold.

(i) $\left(MGP_m(x)MGP_n(x)\right)^T=MGP_n(x)MGP_m(x)$,

(ii) $\left(MGq_m(x)MGq_n(x)\right)^T=MGq_n(x)MGq_m(x)$.

Proof. Since the transpose of the matrix, we obtain

$$\left(MGP_m(x)MGP_n(x)\right)^T=\left(MGP_n(x)\right)^T\left(MGP_m(x)\right)^T=MGP_n(x)MGP_m(x).$$

The similar proof of (i) is used for (ii).

Therefore, the identities (i) and (ii) are easily seen.

Corollary 17 For $m,n\in\mathbb{N}$, the following results hold.

(i) $\left(MGP_mMGP_n\right)^T=MGP_nMGP_m$,

(ii) $\left(MGQ_mMGQ_n\right)^T=MGQ_nMGQ_m$,

(iii) $\left(MGq_mMGq_n\right)^T=MGq_nMGq_m$.

Proof. Take $x=1$ in Theorem 16 (i) and (ii), we have (i) and (iii).

The similar proof of Theorem 16 is used for (ii).

Corollary 18 For $m,n \in \mathbb{N}$, the following results hold.

(i) $\left(MP_m(x)MP_n(x)\right)^T = MP_n(x)MP_m(x)$,

(ii) $\left(MQ_m(x)MQ_n(x)\right)^T = MQ_n(x)MQ_m(x)$,

(iii) $\left(Mq_m(x)Mq_n(x)\right)^T = Mq_n(x)Mq_m(x)$.

Proof. The similar proof of Theorem 16 is used for (i), (ii), and (iii).

Finally, we find some identity matrix sequences of summations by using Binet's formulas (49), (50), (56), (57), (58), (59), (60) and (61) as the following theorem and corollary.

Theorem 19 For $n \in \mathbb{N}$, $x > 0$, and $x^2 + 1 > 0$, the following equalities hold.

(i) $\displaystyle\sum_{k=0}^{n}\frac{1}{t^k}MGP_k(x) = \frac{1}{t^2-2xt-1}\left(t^2MGP_0(x)+tMGP_{-1}(x)\right) - \frac{1}{t^n\left(t^2-2xt-1\right)}\left(tMGP_{n+1}(x)+MGP_n(x)\right)$,

(ii) $\displaystyle\sum_{k=0}^{n}\frac{1}{t^k}MGq_k(x) = \frac{1}{t^2-2xt-1}\left(t^2MGq_0(x)+tMGq_{-1}(x)\right) - \frac{1}{t^n\left(t^2-2xt-1\right)}\left(tMGq_{n+1}(x)+MGq_n(x)\right)$.

Proof. Let $MGP_0(x)$, $MGP_1(x)$ be initial conditions of 2x2 matrix sequence and $\alpha(x) = x+\sqrt{x^2+1}$, $\beta(x) = x-\sqrt{x^2+1}$ be the roots of $t^2-2xt-1=0$.

Then we can write

$$\sum_{k=0}^{n}\frac{1}{t^k}MGP_k(x) = \sum_{k=0}^{n}\left(\frac{\alpha^k(x)}{t^k\left(\alpha(x)-\beta(x)\right)}\left(MGP_1(x)-\beta(x)MGP_0(x)\right) - \frac{\beta^k(x)}{t^k\left(\alpha(x)-\beta(x)\right)}\left(MGP_1(x)-\alpha(x)MGP_0(x)\right)\right). \quad (73)$$

By definition of a geometric sequence, we have

$$\begin{aligned}
&\sum_{k=0}^{n}\left(\frac{\alpha^k(x)}{t^k\left(\alpha(x)-\beta(x)\right)}\left(MGP_1(x)-\beta(x)MGP_0(x)\right) - \frac{\beta^k(x)}{t^k\left(\alpha(x)-\beta(x)\right)}\left(MGP_1(x)-\alpha(x)MGP_0(x)\right)\right)\\
&= \frac{\left(1-\left(\frac{\alpha(x)}{t}\right)^{n+1}\right)}{\left(\alpha(x)-\beta(x)\right)\left(1-\frac{\alpha(x)}{t}\right)}\left(MGP_1(x)-\beta(x)MGP_0(x)\right) - \frac{\left(1-\left(\frac{\beta(x)}{t}\right)^{n+1}\right)}{\left(\alpha(x)-\beta(x)\right)\left(1-\frac{\beta(x)}{t}\right)}\left(MGP_1(x)-\alpha(x)MGP_0(x)\right)\\
&= \frac{t\left(t^{n+1}-\alpha^{n+1}(x)\right)\left(t-\beta(x)\right)}{t^{n+1}\left(\alpha(x)-\beta(x)\right)\left(t-\alpha(x)\right)\left(t-\beta(x)\right)}\left(MGP_1(x)-\beta(x)MGP_0(x)\right)\\
&\quad - \frac{t\left(t^{n+1}-\beta^{n+1}(x)\right)\left(t-\alpha(x)\right)}{t^{n+1}\left(\alpha(x)-\beta(x)\right)\left(t-\beta(x)\right)\left(t-\alpha(x)\right)}\left(MGP_1(x)-\alpha(x)MGP_0(x)\right). \quad (74)
\end{aligned}$$

Since $\left(t-\alpha(x)\right)\left(t-\beta(x)\right) = t^2-2xt-1$, we can write

$$\frac{t\left(t^{n+1}-\alpha^{n+1}(x)\right)\left(t-\beta(x)\right)}{t^{n+1}\left(\alpha(x)-\beta(x)\right)\left(t-\alpha(x)\right)\left(t-\beta(x)\right)}\left(MGP_1(x)-\beta(x)MGP_0(x)\right)$$
$$-\frac{t\left(t^{n+1}-\beta^{n+1}(x)\right)\left(t-\alpha(x)\right)}{t^{n+1}\left(\alpha(x)-\beta(x)\right)\left(t-\beta(x)\right)\left(t-\alpha(x)\right)}\left(MGP_1(x)-\alpha(x)MGP_0(x)\right)$$
$$=\frac{\left(t^{n+1}-\alpha^{n+1}(x)\right)\left(t-\beta(x)\right)}{t^{n}\left(t^2-2xt-1\right)\left(\alpha(x)-\beta(x)\right)}\left(MGP_1(x)-\beta(x)MGP_0(x)\right)$$
$$-\frac{\left(t^{n+1}-\beta^{n+1}(x)\right)\left(t-\alpha(x)\right)}{t^{n}\left(t^2-2xt-1\right)\left(\alpha(x)-\beta(x)\right)}\left(MGP_1(x)-\alpha(x)MGP_0(x)\right). \tag{75}$$

By using (75) in (74), we obtain

$$\sum_{k=0}^{n}\left(\frac{\alpha^k(x)}{t^k\left(\alpha(x)-\beta(x)\right)}\left(MGP_1(x)-\beta(x)MGP_0(x)\right)-\frac{\beta^k(x)}{t^k\left(\alpha(x)-\beta(x)\right)}\left(MGP_1(x)-\alpha(x)MGP_0(x)\right)\right)$$
$$=\frac{\left(t^{n+1}-\alpha^{n+1}(x)\right)\left(t-\beta(x)\right)}{t^{n}\left(t^2-2xt-1\right)\left(\alpha(x)-\beta(x)\right)}\left(MGP_1(x)-\beta(x)MGP_0(x)\right)$$
$$-\frac{\left(t^{n+1}-\beta^{n+1}(x)\right)\left(t-\alpha(x)\right)}{t^{n}\left(t^2-2xt-1\right)\left(\alpha(x)-\beta(x)\right)}\left(MGP_1(x)-\alpha(x)MGP_0(x)\right). \tag{76}$$

By using (76) in (73), we get that

$$\sum_{k=0}^{n}\frac{1}{t^k}MGP_k(x)=\frac{\left(t^{n+1}-\alpha^{n+1}(x)\right)\left(t-\beta(x)\right)}{t^{n}\left(t^2-2xt-1\right)\left(\alpha(x)-\beta(x)\right)}\left(MGP_1(x)-\beta(x)MGP_0(x)\right)$$
$$-\frac{\left(t^{n+1}-\beta^{n+1}(x)\right)\left(t-\alpha(x)\right)}{t^{n}\left(t^2-2xt-1\right)\left(\alpha(x)-\beta(x)\right)}\left(MGP_1(x)-\alpha(x)MGP_0(x)\right)$$
$$=\frac{\left(t^{n+2}-\beta(x)t^{n+1}-\alpha^{n+1}(x)t+\alpha^{n+1}(x)\beta(x)\right)}{t^{n}\left(t^2-2xt-1\right)\left(\alpha(x)-\beta(x)\right)}\left(MGP_1(x)-\beta(x)MGP_0(x)\right)$$
$$-\frac{\left(t^{n+2}-\alpha(x)t^{n+1}-\beta^{n+1}(x)t+\beta^{n+1}(x)\alpha(x)\right)}{t^{n}\left(t^2-2xt-1\right)\left(\alpha(x)-\beta(x)\right)}\left(MGP_1(x)-\alpha(x)MGP_0(x)\right)$$
$$=\frac{1}{t^n\left(t^2-2xt-1\right)}\left(t^{n+2}MGP_0(x)+t^{n+1}MGP_1(x)-\left(\alpha(x)+\beta(x)\right)t^{n+1}MGP_0(x)-tMGP_{n+1}(x)-MGP_n(x)\right)$$
$$=\frac{1}{t^n\left(t^2-2xt-1\right)}\left(t^{n+2}MGP_0(x)+t^{n+1}\left(MGP_1(x)-2xMGP_0(x)\right)-tMGP_{n+1}(x)-MGP_n(x)\right). \tag{77}$$

Take $n=1$ in (33), we can write

$$MGP_1(x)-2xMGP_0(x)=MGP_{-1}(x). \tag{78}$$

By using (78) in (77), we obtain

$$\sum_{k=0}^{n}\frac{1}{t^k}MGP_k(x)=\frac{1}{t^n\left(t^2-2xt-1\right)}\left(t^{n+2}MGP_0(x)+t^{n+1}MGP_{-1}(x)-tMGP_{n+1}(x)-MGP_n(x)\right)$$

$$=\frac{1}{t^2-2xt-1}\left(t^2MGP_0(x)+tMGP_{-1}(x)\right)-\frac{1}{t^n\left(t^2-2xt-1\right)}\left(tMGP_{n+1}(x)+MGP_n(x)\right).$$

The similar proof of (i) is used for (ii).

Therefore, the identities (i) and (ii) are immediately seen.

Corollary 20 For $n\in\mathbb{N}$, $x>0$, and $x^2+1>0$, the following equalities hold.

(i) $\sum_{k=0}^{n}\frac{1}{t^k}MGP_k=\frac{1}{t^2-2t-1}\left(t^2MGP_0+tMGP_{-1}\right)-\frac{1}{t^n\left(t^2-2t-1\right)}\left(tMGP_{n+1}+MGP_n\right)$,

(iii) $\sum_{k=0}^{n}\frac{1}{t^k}MGQ_k=\frac{1}{t^2-2t-1}\left(t^2MGQ_0+tMGQ_{-1}\right)-\frac{1}{t^n\left(t^2-2t-1\right)}\left(tMGQ_{n+1}+MGQ_n\right)$,

(iii) $\sum_{k=0}^{n}\frac{1}{t^k}MGq_k=\frac{1}{t^2-2t-1}\left(t^2MGq_0+tMGq_{-1}\right)-\frac{1}{t^n\left(t^2-2t-1\right)}\left(tMGq_{n+1}+MGq_n\right)$.

Proof. Take $x=1$ in Theorem 19 (i) and (ii), we have (i) and (iii).

The similar proof of Theorem 19 is used for (ii).

Corollary 21 For $n\in\mathbb{N}$, $x>0$, and $x^2+1>0$, the following equalities hold.

(i) $\sum_{k=0}^{n}\frac{1}{t^k}MP_k(x)=\frac{1}{t^2-2xt-1}\left(t^2MP_0(x)+tMP_{-1}(x)\right)-\frac{1}{t^n\left(t^2-2xt-1\right)}\left(tMP_{n+1}(x)+MP_n(x)\right)$,

(ii) $\sum_{k=0}^{n}\frac{1}{t^k}MQ_k(x)=\frac{1}{t^2-2xt-1}\left(t^2MQ_0(x)+tMQ_{-1}(x)\right)-\frac{1}{t^n\left(t^2-2xt-1\right)}\left(tMQ_{n+1}(x)+MQ_n(x)\right)$,

(iii) $\sum_{k=0}^{n}\frac{1}{t^k}Mq_k(x)=\frac{1}{t^2-2xt-1}\left(t^2Mq_0(x)+tMq_{-1}(x)\right)-\frac{1}{t^n\left(t^2-2xt-1\right)}\left(tMq_{n+1}(x)+Mq_n(x)\right)$.

Proof. The similar proof of Theorem 19 is used for (i), (ii), and (iii).

Lemma 22 For $m,\ j\in\mathbb{N}$ and $j\geq m$, the following results hold.

(i) $(-1)^m MGP_{j-m}(x)=\frac{\alpha^j(x)\beta^m(x)}{\alpha(x)-\beta(x)}\left(MGP_1(x)-\beta(x)MGP_0(x)\right)-\frac{\beta^j(x)\alpha^m(x)}{\alpha(x)-\beta(x)}\left(MGP_1(x)-\alpha(x)MGP_0(x)\right)$,

(ii) $(-1)^m MGq_{j-m}(x)=\frac{\alpha^j(x)\beta^m(x)}{\alpha(x)-\beta(x)}\left(MGq_1(x)-\beta(x)MGq_0(x)\right)-\frac{\beta^j(x)\alpha^m(x)}{\alpha(x)-\beta(x)}\left(MGq_1(x)-\alpha(x)MGq_0(x)\right)$,

(iii) $(-1)^m MGP_{j-m}=\frac{\alpha^j\beta^m}{\alpha-\beta}\left(MGP_1-\beta MGP_0\right)-\frac{\beta^j\alpha^m}{\alpha-\beta}\left(MGP_1-\alpha MGP_0\right)$,

(iv) $(-1)^m MGQ_{j-m}=\frac{\alpha^j\beta^m}{\alpha-\beta}\left(MGQ_1-\beta MGQ_0\right)-\frac{\beta^j\alpha^m}{\alpha-\beta}\left(MGQ_1-\alpha MGQ_0\right)$,

(v) $(-1)^m MGq_{j-m}=\frac{\alpha^j\beta^m}{\alpha-\beta}\left(MGq_1-\beta MGq_0\right)-\frac{\beta^j\alpha^m}{\alpha-\beta}\left(MGq_1-\alpha MGq_0\right)$,

(vi) $(-1)^m MP_{j-m}(x)=\frac{\alpha^j(x)\beta^m(x)}{\alpha(x)-\beta(x)}\left(MP_1(x)-\beta(x)MP_0(x)\right)-\frac{\beta^j(x)\alpha^m(x)}{\alpha(x)-\beta(x)}\left(MP_1(x)-\alpha(x)MP_0(x)\right)$,

(vii) $(-1)^m MQ_{j-m}(x)=\dfrac{\alpha^j(x)\beta^m(x)}{\alpha(x)-\beta(x)}\big(MQ_1(x)-\beta(x)MQ_0(x)\big)-\dfrac{\beta^j(x)\alpha^m(x)}{\alpha(x)-\beta(x)}\big(MQ_1(x)-\alpha(x)MQ_0(x)\big)$,

(viii) $(-1)^m Mq_{j-m}(x)=\dfrac{\alpha^j(x)\beta^m(x)}{\alpha(x)-\beta(x)}\big(Mq_1(x)-\beta(x)Mq_0(x)\big)-\dfrac{\beta^j(x)\alpha^m(x)}{\alpha(x)-\beta(x)}\big(Mq_1(x)-\alpha(x)Mq_0(x)\big)$.

Proof. Since (49) and $\alpha(x)=x+\sqrt{x^2+1}$, $\beta(x)=x-\sqrt{x^2+1}$ be the roots of $t^2-2xt-1=0$, we obtain

$$\begin{aligned}(-1)^m MGP_{j-m}(x)&=(-1)^m\frac{\alpha^{j-m}(x)}{\alpha(x)-\beta(x)}\big(MGP_1(x)-\beta(x)MGP_0(x)\big)-(-1)^m\frac{\beta^{j-m}(x)}{\alpha(x)-\beta(x)}\big(MGP_1(x)-\alpha(x)MGP_0(x)\big)\\&=(-1)^m\frac{\alpha^j(x)\beta^m(x)}{\big(\alpha(x)-\beta(x)\big)\alpha^m(x)\beta^m(x)}\big(MGP_1(x)-\beta(x)MGP_0(x)\big)\\&\quad-(-1)^m\frac{\beta^j(x)\alpha^m(x)}{\big(\alpha(x)-\beta(x)\big)\beta^m(x)\alpha^m(x)}\big(MGP_1(x)-\alpha(x)MGP_0(x)\big)\\&=\frac{\alpha^j(x)\beta^m(x)}{\alpha(x)-\beta(x)}\big(MGP_1(x)-\beta(x)MGP_0(x)\big)-\frac{\beta^j(x)\alpha^m(x)}{\alpha(x)-\beta(x)}\big(MGP_1(x)-\alpha(x)MGP_0(x)\big).\end{aligned}$$

Thus, $(-1)^m MGP_{j-m}(x)=\dfrac{\alpha^j(x)\beta^m(x)}{\alpha(x)-\beta(x)}\big(MGP_1(x)-\beta(x)MGP_0(x)\big)-\dfrac{\beta^j(x)\alpha^m(x)}{\alpha(x)-\beta(x)}\big(MGP_1(x)-\alpha(x)MGP_0(x)\big)$.

The similar proof of (i) is used for (ii), (iii), (iv), (v), (vi), (vii), and (viii).

Therefore, the identities (i), (ii), (iii), (iv), (v), (vi), (vii), and (viii) are immediately seen.

Theorem 23 For $m,n,j\in\mathbb{N}$ and $j\geq m$, the following results hold.

(i)

$$\sum_{k=0}^{n}MGP_{mk+j}(x)=\frac{1}{\big(1-\alpha^m(x)\big)\big(1-\beta^m(x)\big)}\Big(MGP_j(x)+(-1)^{m+1}MGP_{j-m}(x)-MGP_{m\,n+m+j}(x)+(-1)^m MGP_{m\,n+j}(x)\Big),$$

(ii)

$$\sum_{k=0}^{n}MGq_{mk+j}(x)=\frac{1}{\big(1-\alpha^m(x)\big)\big(1-\beta^m(x)\big)}\Big(MGq_j(x)+(-1)^{m+1}MGq_{j-m}(x)-MGq_{m\,n+m+j}(x)+(-1)^m MGq_{m\,n+j}(x)\Big).$$

Proof. Let $\alpha(x)=x+\sqrt{x^2+1}$, $\beta(x)=x-\sqrt{x^2+1}$ be the roots of $t^2-2xt-1=0$. Then we have

$$\sum_{k=0}^{n}MGP_{m\,k+j}(x)$$

$$=\sum_{k=0}^{n}\left(\frac{\alpha^{m\,k+j}(x)}{\alpha(x)-\beta(x)}\big(MGP_1(x)-\beta(x)MGP_0(x)\big)-\frac{\beta^{m\,k+j}(x)}{\alpha(x)-\beta(x)}\big(MGP_1(x)-\alpha(x)MGP_0(x)\big)\right). \quad (79)$$

Since definition of a geometric sequence, we have

$$\sum_{k=0}^{n}\left(\frac{\alpha^{m\,k+j}(x)}{\alpha(x)-\beta(x)}\big(MGP_1(x)-\beta(x)MGP_0(x)\big)-\frac{\beta^{m\,k+j}(x)}{\alpha(x)-\beta(x)}\big(MGP_1(x)-\alpha(x)MGP_0(x)\big)\right)$$

$$=\frac{\alpha^{j}(x)\left(1-\alpha^{m\,n+m}(x)\right)}{\left(\alpha(x)-\beta(x)\right)\left(1-\alpha^{m}(x)\right)}\left(MGP_1(x)-\beta(x)MGP_0(x)\right)-\frac{\beta^{j}(x)\left(1-\beta^{m\,n+m}(x)\right)}{\left(\alpha(x)-\beta(x)\right)\left(1-\beta^{m}(x)\right)}\left(MGP_1(x)-\alpha(x)MGP_0(x)\right)$$

$$=\frac{\alpha^{j}(x)\left(1-\alpha^{m\,n+m}(x)\right)\left(1-\beta^{m}(x)\right)}{\left(\alpha(x)-\beta(x)\right)\left(1-\alpha^{m}(x)\right)\left(1-\beta^{m}(x)\right)}\left(MGP_1(x)-\beta(x)MGP_0(x)\right)$$

$$-\frac{\beta^{j}(x)\left(1-\beta^{m\,n+m}(x)\right)\left(1-\alpha^{m}(x)\right)}{\left(\alpha(x)-\beta(x)\right)\left(1-\beta^{m}(x)\right)\left(1-\alpha^{m}(x)\right)}\left(MGP_1(x)-\alpha(x)MGP_0(x)\right). \quad (80)$$

By using Lemma 22 (i) in (80), we have

$$\sum_{k=0}^{n}\left(\frac{\alpha^{m\,k+j}(x)}{\alpha(x)-\beta(x)}\left(MGP_1(x)-\beta(x)MGP_0(x)\right)-\frac{\beta^{m\,k+j}(x)}{\alpha(x)-\beta(x)}\left(MGP_1(x)-\alpha(x)MGP_0(x)\right)\right)$$

$$=\frac{\alpha^{j}(x)\left(1-\alpha^{m\,n+m}(x)\right)\left(1-\beta^{m}(x)\right)}{\left(\alpha(x)-\beta(x)\right)\left(1-\alpha^{m}(x)\right)\left(1-\beta^{m}(x)\right)}\left(MGP_1(x)-\beta(x)MGP_0(x)\right)$$

$$-\frac{\beta^{j}(x)\left(1-\beta^{m\,n+m}(x)\right)\left(1-\alpha^{m}(x)\right)}{\left(\alpha(x)-\beta(x)\right)\left(1-\beta^{m}(x)\right)\left(1-\alpha^{m}(x)\right)}\left(MGP_1(x)-\alpha(x)MGP_0(x)\right)$$

$$=\frac{1}{\left(1-\alpha^{m}(x)\right)\left(1-\beta^{m}(x)\right)}\left(MGP_j(x)+(-1)^{m+1}MGP_{j-m}(x)-MGP_{m\,n+m+j}(x)+(-1)^{m}MGP_{m\,n+j}(x)\right).$$

Thus,

$$\sum_{k=0}^{n}MGP_{mk+j}(x)=\frac{1}{\left(1-\alpha^{m}(x)\right)\left(1-\beta^{m}(x)\right)}\left(MGP_j(x)+(-1)^{m+1}MGP_{j-m}(x)-MGP_{m\,n+m+j}(x)+(-1)^{m}MGP_{m\,n+j}(x)\right).$$

The similar proof of (i) is used for (ii).

Therefore, the proof is complete.

Corollary 24 For $m,n,j\in\mathbb{N}$ and $j\geq m$, the following results hold.

(i) $\sum_{k=0}^{n}MGP_{mk+j}=\frac{1}{\left(1-\alpha^{m}\right)\left(1-\beta^{m}\right)}\left(MGP_j+(-1)^{m+1}MGP_{j-m}-MGP_{m\,n+m+j}+(-1)^{m}MGP_{m\,n+j}\right)$,

(ii) $\sum_{k=0}^{n}MGQ_{mk+j}=\frac{1}{\left(1-\alpha^{m}\right)\left(1-\beta^{m}\right)}\left(MGQ_j+(-1)^{m+1}MGQ_{j-m}-MGQ_{m\,n+m+j}+(-1)^{m}MGQ_{m\,n+j}\right)$,

(iii) $\sum_{k=0}^{n}MGq_{mk+j}=\frac{1}{\left(1-\alpha^{m}\right)\left(1-\beta^{m}\right)}\left(MGq_j+(-1)^{m+1}MGq_{j-m}-MGq_{m\,n+m+j}+(-1)^{m}MGq_{m\,n+j}\right)$.

Proof. Take $x=1$ in Theorem 23 (i) and (ii), we have (i) and (iii).

The similar proof of Theorem 23 is used for (ii).

Corollary 25 For $m,n,j\in\mathbb{N}$ and $j\geq m$, the following results hold.

(i) $\sum_{k=0}^{n}MP_{mk+j}(x)=\frac{1}{\left(1-\alpha^{m}(x)\right)\left(1-\beta^{m}(x)\right)}\left(MP_j(x)+(-1)^{m+1}MP_{j-m}(x)-MP_{m\,n+m+j}(x)+(-1)^{m}MP_{m\,n+j}(x)\right)$,

(ii) $\sum_{k=0}^{n} MQ_{mk+j}(x) = \frac{1}{(1-\alpha^m(x))(1-\beta^m(x))}\left(MQ_j(x) + (-1)^{m+1} MQ_{j-m}(x) - MQ_{m\,n+m+j}(x) + (-1)^m MQ_{m\,n+j}(x)\right),$

(iii) $\sum_{k=0}^{n} Mq_{mk+j}(x) = \frac{1}{(1-\alpha^m(x))(1-\beta^m(x))}\left(Mq_j(x) + (-1)^{m+1} Mq_{j-m}(x) - Mq_{m\,n+m+j}(x) + (-1)^m Mq_{m\,n+j}(x)\right).$

Proof. The similar proof of Theorem 23 is used for (i), (ii), and (iii).

Discussion

In this article, we get some identities of the relation between matrix sequences and summations by applying some properties of matrix operation, the relation between numbers and polynomials, and Binet's formulas of matrix sequences.

Conclusions

In this paper, some identities of matrix sequences prove by some properties of matrix operation, the relation between numbers and polynomials, and Binet's formulas of 2 x 2 matrix representation. We obtained especially some identities of the relationships between matrix sequences. Moreover, we conjecture which this concept extends to the matrix sequence in terms of other recurrence relations and present the n x n matrix for $n \geq 3$.

Acknowledgements

This research was supported by the Faculty of Science and Technology, Rajamangala University of Technology Thanyaburi (RMUTT), Pathum Thani, THAILAND.

References

Daykin, D.E. & Dresel, L.A.G. (1967). Identities for Products of Fibonacci and Lucas Numbers. *The Fibonacci Quarterly*, *5*(4), 367 – 370.

Gulec, H.H. & Taskara, N. (2012). On the (s,t)-Pell and (s,t)- Pell-Lucas Sequences and Their Matrix Representations. *Applied Mathematics Letters*, *25*(10), 1554 – 1559.

Halici, S. & Öz, S. (2016). On Some Gaussian Pell and Pell-Lucas Numbers. *Ordu University Journal of Science and Tecnology*, *6*(1), 8 – 18.

Halici, S. & Oz, S. (2018). On Gaussian Pell Polynomials and Their Some Properties. *Palestine Joumal of mathematies*, *7*(1), 251 – 256.

Horadam, A. F. (1961). A Generalized Fibonacci Sequence. *The American Mathematical Monthly*, *68*(5), p. 455 – 459.

Horadam, A.F. & Mahon, Bro.J.M. (1985). Pell and Pell-Lucas Polynomials. *The Fibonacci Quarterly* , *23*(1), 7 – 20.

Horadam, A.F. (1984). Pell Numbers and Coaxal Circles. *The Fibonacci Quarterly* , *22*(4), 324 – 326.

Karaaslan, N. (2019). A Note on Modified Pell Polynomials. *Aksaray University Journal of Science and Engineering*, *3*(1), 1 – 7.

Yagmur, T. & Karaaslan, N. (2018). Gaussian Modified Pell Sequence and Gaussian Modified Pell Polynomial Sequence. *Aksaray University Journal of Science and Engineering*, *2*(1), 63 – 72.