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# เซตขอบ เซตภายนอก และเซตหนาแน่นในปริภูมิสองโครงสร้างอ่อน

## Boundary Sets, Exterior Sets and Dense Sets in Bi-weak Structure Spaces

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## บทคัดย่อ

ในบทความนี้ จะนำเสนอแนวคิดของเซตขอบ เซตภายนอก และเซตหนาแน่นในปริภูมิสองโครงสร้างอ่อน ได้แสดงให้เห็นสมบัติบางประการของเซตเหล่านี้ โดยเฉพาะอย่างยิ่ง ได้รับบางลักษณะเฉพาะของเซตปิดในปริภูมิสองโครงสร้างอ่อนโดยใช้เซตขอบ หรือเซตภายนอก

**คำสำคัญ** : เซตขอบ, เซตหนาแน่น, เซตภายนอก, ปริภูมิสองโครงสร้างอ่อน

## Abstract

In this article, the concepts of boundary sets, exterior sets and dense sets in bi-weak structure spaces are introduced. Some properties of their sets are obtained. In particular, some characterizations of closed sets in bi-weak structure spaces using boundary sets or exterior sets are obtained.

**Keywords:** boundary set, exterior set, dense set, bi-weak structure space.

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## Introduction

The notion of bitopological spaces, which consists of a set and two topologies, and some separation axioms in a bitopological space were introduced by Kelly (1963). Later, Popa and Noiri (2000), studied the concept of minimal structures. Csa´sza´r (2002, 2011), introduced the notions of generalized topologies and weak structures. These structures are generalizations of topologies. Next, the concepts of bigeneralized topological spaces and biminimal structure spaces were introduced by Boonpok (2010). Later, Sompong (2011), studied some properties of boundary sets and exterior sets in biminimal structure spaces, respectively. Moreover, Sompong (2012), studied some properties of dense sets in biminimal structure spaces. Recently, Puiwong *et al.,* (2017), introduced the concept of bi-weak structure spaces or briefly bi-w spaces and studied the properties of closed sets and some separation axioms in bi-weak structure spaces.

In this article, we will extend the concepts of boundary sets, exterior sets and dense sets in bi-weak structure spaces and study some fundamental of their properties.

## Methods

In this research, we shall use the methods of proof in mathematics and the basic concepts of weak structures and bi-weak structure spaces. Now, we recall about some properties of weak structures and bi-weak structure spaces.

Definition 2.1. [Csa´sza´r (2011)]. Let $X$ be a nonempty set and $P(X)$ the power set of $X$. A subfamily $w$ of $P(X)$ is called a *weak structure* (briefly WS) on $X$ if $\emptyset \in w$.

By $(X, w)$ we denote a nonempty set $X$ with a WS $w$ on $X$ and it is called a *w-space*. The elements of $w$ are called *w-open sets* and the complements are called *w-closed sets*.

Let $w$ be a weak structure on $X$ and $A \subseteq X$, the w-closure of $A$, denoted by $c_w(A)$ and w-interior of $A$, denoted by $i_w(A)$. We define $c_w(A)$ as the intersection of all w-closed sets containing $A$ and $i_w(A)$ as the union of all w-open subsets of $A$.

Theorem 2.2. [Csa´sza´r (2011)]. Let $w$ be a $WS$ on $X$ and $A, B \subseteq X$. Then

1. $A \subseteq c_w(A)$ and $i_w(A) \subseteq A$;
2. If $A \subseteq B$, then $c_w(A) \subseteq c_w(B)$ and $i_w(A) \subseteq i_w(B)$;
3. $c_w(c_w(A)) = c_w(A)$ and $i_w(i_w(A)) = i_w(A)$;
4. $c_w(X \setminus A) = X \setminus i_w(A)$ and $i_w(X \setminus A) = X \setminus c_w(A)$;
5. $x \in i_w(A)$ if and only if there is a w-open set $V$ such that $x \in V \subseteq A$;
6. $x \in c_w(A)$ if and only if $V \cap A \neq \emptyset$ for any w-open set $V$ containing;
7. If $A \in w$, then $A = i_w(A)$. And if $A$ is w-closed, then $A = c_w(A)$.

*Definition 2.3*. [Puiwong et al., (2017)]. Let $X$ be a nonempty set and $w^1$, $w^2$ be two weak structures on $X$. A triple $(X, w^1, w^2)$ is called a *bi-weak structure space* (briefly *bi-w space*).

Let $(X, w^1, w^2)$ be a bi-w space and $A$ be a subset of $X$. The w-closure and w-interior of $A$ with respect to $w^i$ are denoted by $c_{w^i}(A)$ and $i_{w^i}(A)$ where $i \in \{1, 2\}$.

*Definition 2.4.* [Puiwong *et al.,* (2017)]. A subset $A$ of a bi-w space $(X, w^1, w^2)$ is call *closed* if $A = c_{w^1}(c_{w^2}(A))$. The complement of a closed set is called *open*.

*Remark* In this paper, we shall call closed and open in a bi-w space that *bi-w-closed* and *bi-w-open*, respectively.

*Theorem 2.5.* [Puiwong *et al.,* (2017)]. Let $(X, w^1, w^2)$ be a bi-w space and $A$ be a subset of $X$. Then the following are equivalent:

1. $A$ is bi-w-closed;
2. $A = c_{w^1}(A)$ and $A = c_{w^2}(A)$;
3. $A = c_{w^2}(c_{w^1}(A))$.

*Proposition 2.6.* [Puiwong *et al.*, (2017)]. Let $(X, w^1, w^2)$ be a bi-w space and $A \subseteq X$. If $A$ is both w-closed respect to $w^1$ and $w^2$, then $A$ is a bi-w-closed set in the bi-w space $(X, w^1, w^2)$.

*Proposition 2.7*. [Puiwong *et al.*, (2017)]. Let $(X, w^1, w^2)$ be a bi-w space. If $A_\alpha$ is bi-w-closed for all $\alpha \in \Lambda \neq \emptyset$, then $\cap_{\alpha\in\Lambda} A_\alpha$ is bi-w-closed.

*Proposition 2.8.* [Puiwong *et al.*, (2017)]. Let $(X, w^1, w^2)$ be a bi-w space. If $A_\alpha$ is bi-$w$-open for all $\alpha \in \Lambda \neq \emptyset$, then $\cup_{\alpha\in\Lambda} A_\alpha$ is bi-w-open.

*Theorem 2.9.* [Puiwong *et al.,* (2017)]. Let $(X, w^1, w^2)$ be a bi-w space and $A$ be a subset of $X$. Then the following are equivalent:

1. $A$ is bi-w-open;
2. $A = i_{w^1}(i_{w^2}(A))$;
3. $A = i_{w^1}(A)$ and $A = i_{w^2}(A)$;
4. $A = i_{w^2}(i_{w^1}(A))$.

## Results

In this section, we introduce the concepts of boundary sets, exterior sets and dense sets in bi-weak structure spaces and study some fundamental of their properties. Next, let $i, j \in \{1, 2\}$ be such that $i \neq j$.

*3.1 Boundary sets in bi-weak structure spaces*

*Definition 3.1.1.* Let $(X, w^1, w^2)$ be a bi-w space, $A$ be a subset of $X$ and $x \in X$. We called $x$ is a $w^i w^j$*-boundary point of* $A$ if $x \in c_{w^i}(c_{w^j}(A)) \cap c_{w^i}(c_{w^j}(X \setminus A))$. We denote the set of all $w^i w^j$-boundary points of $A$ by $wBdr_{ij}(A)$.

Remark From the above definition, it is easy to verify that $wBdr_{ij}(A) = c_{w^i}(c_{w^j}(A)) \cap c_{w^i}(c_{w^j}(X\backslash A))$.

Example 3.1.2. Let $X = \{1,2,3\}$. Define weak structures $w^1$ and $w^2$ on $X$ as follows: $w^1 = \{\emptyset, \{1\}, \{2,3\}\}$ and $w^2 = \{\emptyset, \{3\}, \{1,2\}\}$. Hence $wBdr_{12}(\{1\}) = X$ and $wBdr_{21}(\{1\}) = \{1,2\}$.

Lemma 3.1.3. Let $(X, w^1, w^2)$ be a bi-w space and $A$ be a subset of $X$. Then $wBdr_{ij}(X\backslash A) = wBdr_{ij}(A)$.

Proof. Since $wBdr_{ij}(X\backslash A) = c_{w^i}(c_{w^j}(X\backslash A)) \cap c_{w^i}(c_{w^j}(X(X\backslash A)))$ and $wBdr_{ij}(A) = c_{w^i}(c_{w^j}(A)) \cap c_{w^i}(c_{w^j}(X\backslash A))$, $wBdr_{ij}(X\backslash A) = wBdr_{ij}(A)$.

Theorem 3.1.4. Let $(X, w^1, w^2)$ be a bi-w space and $A \subseteq X$. Then the following statements hold;

1. $wBdr_{ij}(A) = c_{w^i}(c_{w^j}(A))\backslash i_{w^i}(i_{w^j}(A))$;
2. $wBdr_{ij}(A) \cap i_{w^i}(i_{w^j}(A)) = \emptyset$;
3. $wBdr_{ij}(A) \cap i_{w^i}(i_{w^j}(X\backslash A)) = \emptyset$;
4. $c_{w^i}(c_{w^j}(A)) = wBdr_{ij}(A) \cup i_{w^i}(i_{w^j}(A))$
5. $X = i_{w^i}(i_{w^j}(A)) \cup wBdr_{ij}(A) \cup i_{w^i}(i_{w^j}(X\backslash A))$ is a pairwise disjoint union;
6. $c_{w^i}(c_{w^j}(A)) = wBdr_{ij}(A) \cup A$.

Proof.

1.
$$\begin{aligned} wBdr_{ij}(A) &= c_{w^i}(c_{w^j}(A)) \cap c_{w^i}(c_{w^j}(X\backslash A)) \\ &= c_{w^i}(c_{w^j}(A)) \cap c_{w^i}(X\backslash i_{w^j}(A)) \\ &= c_{w^i}(c_{w^j}(A)) \cap X\backslash i_{w^i}(i_{w^j}(A)) \\ &= c_{w^i}(c_{w^j}(A))\backslash i_{w^i}(i_{w^j}(A). \end{aligned}$$

2. From 1., we obtain that

$$wBdr_{ij}(A) \cap i_{w^i}(i_{w^j}(A)) = [c_{w^i}(c_{w^j}(A))\backslash i_{w^i}(i_{w^j}(A))] \cap i_{w^i}(i_{w^j}(A)) = \emptyset.$$

3.
$$\begin{aligned} wBdr_{ij}(A) \cap i_{w^i}(i_{w^j}(X\backslash A)) &= [c_{w^i}(c_{w^j}(A)) \cap c_{w^i}(c_{w^j}(X\backslash A))] \cap i_{w^i}(i_{w^j}(X\backslash A)) \\ &= c_{w^i}(c_{w^j}(A)) \cap c_{w^i}(c_{w^j}(X\backslash A)) \cap (X\backslash c_{w^i}(c_{w^j}(A))) \\ &= \emptyset. \end{aligned}$$

4.
$$\begin{aligned} wBdr_{ij}(A) \cup i_{w^i}(i_{w^j}(A)) &= [c_{w^i}(c_{w^j}(A))\backslash i_{w^i}(i_{w^j}(A))] \cup i_{w^i}(i_{w^j}(A)) \\ &= c_{w^i}(c_{w^j}(A)) \cup i_{w^i}(i_{w^j}(A)) \\ &= c_{w^i}(c_{w^j}(A)). \end{aligned}$$

5.
$$\begin{aligned} i_{w^i}(i_{w^j}(A)) \cup wBdr_{ij}(A) \cup i_{w^i}(i_{w^j}(X\backslash A)) &= c_{w^i}(c_{w^j}(A)) \cup i_{w^i}(i_{w^j}(X\backslash A)) \\ &= c_{w^i}(c_{w^j}(A)) \cup i_{w^i}(X\backslash c_{w^j}(A)) \\ &= c_{w^i}(c_{w^j}(A)) \cup X\backslash c_{w^i}(c_{w^j}(A)) \\ &= X. \end{aligned}$$

By 2. and 3., we have $wBdr_{ij}(A) \cap i_{w^i}(i_{w^j}(A)) = \emptyset$ and $wBdr_{ij}(A) \cap i_{w^i}(i_{w^j}(X\backslash A)) = \emptyset$. Now, we will show that $i_{w^i}(i_{w^j}(A)) \cap i_{w^i}(i_{w^j}(X\backslash A)) = \emptyset$. Since $i_{w^i}(i_{w^j}(A)) \subseteq A$ and $i_{w^i}(i_{w^j}(X\backslash A)) \subseteq X\backslash A$, we also have

$i_{w^i}(i_{w^j}(A)) \cap i_{w^i}(i_{w^j}(X\backslash A)) = \emptyset$. Therefore $X = i_{w^i}(i_{w^j}(A)) \cup wBdr_{ij}(A) \cup i_{w^i}(i_{w^j}(X\backslash A))$ is a pairwise disjoint union.

6.
$$\begin{aligned} wBdr_{ij}(A) \cup A &= [c_{w^i}(c_{w^j}(A)) \cap c_{w^i}(c_{w^j}(X\backslash A))] \cup A \\ &= [c_{w^i}(c_{w^j}(A)) \cup A] \cap [c_{w^i}(c_{w^j}(X\backslash A)) \cup A] \\ &= c_{w^i}(c_{w^j}(A)) \cap [c_{w^i}(X\backslash i_{w^j}(A)) \cup A] \\ &= c_{w^i}(c_{w^j}(A)) \cap [X\backslash i_{w^i}(i_{w^j}(A)) \cup A] \\ &= c_{w^i}(c_{w^j}(A)) \cap X \\ &= c_{w^i}(c_{w^j}(A)). \end{aligned}$$

*Theorem 3.1.5.* Let $(X, w^1, w^2)$ be a bi-w space and $A \subseteq X$. Then

1. $A$ is bi-w-closed if and only if $wBdr_{ij}(A) \subseteq A$ .
2. $A$ is bi-w-open if and only if $wBdr_{ij}(A) \subseteq X\backslash A$.

*Proof. 1*. ($\Rightarrow$) Assume that $A$ is bi-w-closed. Thus $c_{w^i}(c_{w^j}(A)) = A$, and so $wBdr_{ij}(A) \cap (X\backslash A) = c_{w^i}(c_{w^j}(A)) \cap c_{w^i}(c_{w^j}(X\backslash A)) \cap (X\backslash A) = A \cap c_{w^i}(c_{w^j}(X\backslash A)) \cap (X\backslash A) = \emptyset$. Therefore $wBdr_{ij}(A) \subseteq A$.

($\Leftarrow$) Assume that $wBdr_{ij}(A) \subseteq A$. Thus $wBdr_{ij}(A) \cap (X\backslash A) = \emptyset$, and so $c_{w^i}\left(c_{w^j}(A)\right) \cap c_{w^i}\left(c_{w^j}(X\backslash A)\right) \cap (X\backslash A) = \emptyset$. Since $X\backslash A \subseteq c_{w^i}(c_{w^j}(X\backslash A))$, we have $c_{w^i}(c_{w^j}(A)) \cap (X\backslash A) = \emptyset$. Then $c_{w^i}(c_{w^j}(A)) \subseteq A$. Clearly, $A \subseteq c_{w^i}(c_{w^j}(A))$. Consequently $A = c_{w^i}(c_{w^j}(A))$. Hence $A$ is bi-w-closed.

2. ($\Rightarrow$) Assume that $A$ is bi-w-open. Thus $i_{w^i}(i_{w^j}(A)) = A$, and so $wBdr_{ij}(A) \cap A = c_{w^i}(c_{w^j}(A)) \cap c_{w^i}(c_{w^j}(X\backslash A)) \cap A = c_{w^i}(c_{w^j}(A)) \cap (X\backslash i_{w^i}(i_{w^j}(A))) \cap A = c_{w^i}(c_{w^j}(A)) \cap (X\backslash A) \cap A = \emptyset$.

Therefore $wBdr_{ij}(A) \subseteq X\backslash A$.

($\Leftarrow$) Assume that $wBdr_{ij}(A) \subseteq X\backslash A$. Thus $wBdr_{ij}(A) \cap A = \emptyset$, and so $c_{w^i}\left(c_{w^j}(A)\right) \cap c_{w^i}\left(c_{w^j}(X\backslash A)\right) \cap A = \emptyset$. Then $c_{w^i}(c_{w^j}(A)) \cap (X\backslash i_{w^i}(i_{w^j}(A))) \cap A = \emptyset$. Since $A \subseteq c_{w^i}(c_{w^j}(A))$, we have $(X\backslash i_{w^i}(i_{w^j}(A))) \cap A = \emptyset$. Thus $A \subseteq i_{w^i}(i_{w^j}(A))$. Clearly, $i_{w^i}(i_{w^j}(A)) \subseteq A$. Hence $A = i_{w^i}(i_{w^j}(A))$. Consequently $A$ is bi-w-open.

*Corollary 3.1.6.* Let $(X, w^1, w^2)$ be a bi-w space and $A$ be a subset of $X$. Then $wBdr_{ij}(A) = \emptyset$ if and only if $A$ is bi-w-closed and bi-w-open.

*Proof.* ($\Rightarrow$) Assume that $wBdr_{ij}(A) = \emptyset$. Thus, we have $wBdr_{ij}(A) \subseteq A$ and $wBdr_{ij}(A) \subseteq X\backslash A$.

By Theorem 3.1.5, we have $A$ is bi-w-closed and bi-w-open.

($\Leftarrow$) Assume that $A$ is bi-w-closed and bi-w-open. By Theorem 3.1.5, we have $wBdr_{ij}(A) \subseteq A$ and $wBdr_{ij}(A) \subseteq X\backslash A$. Therefore $wBdr_{ij}(A) \subseteq A \cap (X\backslash A) = \emptyset$. Hence $wBdr_{ij}(A) = \emptyset$.

*3.2 Exterior sets in bi-weak structure spaces*

*Definition 3.2.1*. Let $(X, w^1, w^2)$ be a bi-w space, $A$ be a subset of $X$ and $x \in X$. We called $x$ is a $w^i w^j$-*exterior point of* $A$ if $x \in i_{w^i}(i_{w^j}(X\backslash A))$. We denote the set of all $w^i w^j$-exterior points of $A$ by $wExt_{ij}(A)$.

*Remark* From the previous definition, it is easy to verify that $wExt_{ij}(A) = X\backslash c_{w^i}(c_{w^j}(A))$.

Example 3.2.2. Let $X=\{1,2,3\}$. Define weak structures $w^1$ and $w^2$ on $X$ as follows: $w^1=\{\emptyset,\{1\},\{2,3\}\}$ and $w^2=\{\emptyset,\{1\},\{1,2\}\}$. Hence $wExt_{12}(\{2\})=X\backslash c_{w^1}(c_{w^2}(\{2\}))=\{1\}$ and $wExt_{21}(\{2\})=X\backslash c_{w^2}(c_{w^1}(\{2\}))=\{1\}$. Moreover, $wExt_{12}(\{\emptyset\})=X\backslash c_{w^1}(c_{w^2}(\{\emptyset\}))=\{1\}$ and $wExt_{21}(\{\emptyset\})=X\backslash c_{w^2}(c_{w^1}(\{\emptyset\}))=\{2,3\}$.

Lemma 3.2.3. Let $(X,w^1,w^2)$ be a bi-w space and $A\subseteq X$. Then

1. $wExt_{ij}(A)\cap A=\emptyset$.
2. $wExt_{ij}(X)=\emptyset$.

Proof. 1. Since $A\subseteq c_{w^i}(c_{w^j}(A))$, $(X\backslash c_{w^i}(c_{w^j}(A)))\cap A\subseteq(X\backslash A)\cap A=\emptyset$. From $wExt_{ij}(A)=X\backslash c_{w^i}(c_{w^j}(A))$, we have $wExt_{ij}(A)\cap A=\emptyset$.

2. From 1. and $wExt_{ij}(X)\subseteq X$ we have $wExt_{ij}(X)=wExt_{ij}(X)\cap X=\emptyset$.

Theorem 3.2.4. Let $(X,w^1,w^2)$ be a bi-w space and $A,B$ be two subsets of $X$. If $A\subseteq B$, then $wExt_{ij}(B)\subseteq wExt_{ij}(A)$.

Proof. Assume that $A\subseteq B$. Thus $c_{w^i}(c_{w^j}(A))\subseteq c_{w^i}(c_{w^j}(B))$ and so $X\backslash c_{w^i}(c_{w^j}(B))\subseteq X\backslash c_{w^i}(c_{w^j}(A))$.

Hence $wExt_{ij}(B)\subseteq wExt_{ij}(A)$.

Theorem 3.2.5. Let $(X,w^1,w^2)$ be a bi-w space and $A$ be a subset of $X$. Then $A$ is bi-w-closed if and only if $wExt_{ij}(A)=X\backslash A$.

Proof. $(\Rightarrow)$ Assume that $A$ is bi-w-closed. Then $A=c_{w^i}(c_{w^j}(A))$. Since $wExt_{ij}(A)=X\backslash c_{w^i}(c_{w^j}(A))=X\backslash A$. Therefore $wExt_{ij}(A)=X\backslash A$.

$(\Leftarrow)$ Assume that $wExt_{ij}(A)=X\backslash A$. Thus $X\backslash c_{w^i}(c_{w^j}(A))=X\backslash A$. Consequently $c_{w^i}(c_{w^j}(A))=A$. Hence $A$ is bi-w-closed.

Corollary 3.2.6. Let $(X,w^1,w^2)$ be a bi-w space and $A$ be a subset of $X$. Then $A$ is bi-w-open if and only if $wExt_{ij}(X\backslash A)=A$.

Proof. It follows from Theorem 3.2.5.

Corollary 3.2.7. Let $(X,w^1,w^2)$ be a bi-w space and $A\subseteq X$. If $A$ is bi-w-closed, then $wExt_{ij}(X\backslash wExt_{ij}(A))=wExt_{ij}(A)$.

Proof. Assume that $A$ is bi-w-closed. From Theorem 3.2.5, $wExt_{ij}(A)=X\backslash A$. Then $A=X\backslash wExt_{ij}(A)$.

Hence $wExt_{ij}(X\backslash wExt_{ij}(A))=wExt_{ij}(A)$.

Theorem 3.2.8. Let $(X,w^1,w^2)$ be a bi-w space and $A,B$ be two subsets of $X$. Then;

1. $wExt_{ij}(A)\cup wExt_{ij}(B)\subseteq wExt_{ij}(A\cap B)$.
2. If $A$ and $B$ are bi-w-closed, then $wExt_{ij}(A)\cup wExt_{ij}(B)=wExt_{ij}(A\cap B)$

Proof. 1. Since $A\cap B\subseteq A$ and $A\cap B\subseteq B$, by Theorem 3.2.4, we have $wExt_{ij}(A)\subseteq wExt_{ij}(A\cap B)$ and $wExt_{ij}(B)\subseteq wExt_{ij}(A\cap B)$. It follows that $wExt_{ij}(A)\cup wExt_{ij}(B)\subseteq wExt_{ij}(A\cap B)$.

2. Assume that $A$ and $B$ are bi-w-closed. Then $A \cap B$ is bi-w-closed. By Theorem 3.2.5, Thus $wExt_{ij}(A \cap B) = X\backslash(A \cap B) = (X\backslash A) \cup (X\backslash B) = wExt_{ij}(A) \cup wExt_{ij}(B)$.

Remark In Theorem 3.2.8, 2. is not true if $A$ and $B$ are not bi-w-closed. We can be seen from the following example.

Example 3.2.9. Let $X = \{1,2,3\}$. Define weak structures $w^1$ and $w^2$ on $X$ as follows: $w^1 = \{\emptyset, \{1\}, \{2,3\}\}$ and $w^2 = \{\emptyset, \{2\}, \{1,3\}\}$. Hence $wExt_{12}(\{1\} \cap \{2\}) = X$, and $wExt_{12}(\{1\}) = X\backslash c_{w^1}(c_{w^2}(\{1\})) = \emptyset$ and $wExt_{12}(\{2\}) = X\backslash c_{w^1}(c_{w^2}(\{2\})) = \{1\}$. Therefore $wExt_{12}(\{1\}) \cup wExt_{12}(\{2\}) \neq wExt_{12}(\{1\} \cap \{2\})$.

3.3 Dense sets in bi-weak structure spaces

Definition 3.3.1. Let $(X, w^1, w^2)$ be a bi-w space. A subset $A$ of $X$ is called a $w^iw^j$*-dense set* in $X$ if $X = c_{w^i}(c_{w^j}(A))$.

Example 3.3.2. Let $X = \{1,2,3\}$. Define weak structures $w^1$ and $w^2$ on $X$ as follows: $w^1 = \{\emptyset, \{1,2\}, \{1,3\}, \{2,3\}\}$ and $w^2 = \{\emptyset, \{1\}, \{3\}, \{2,3\}\}$. Then $c_{w^1}(c_{w^2}(\{3\})) = X$ and $c_{w^2}(c_{w^1}(\{3\})) = \{2,3\}$. Hence $\{3\}$ is a $w^1w^2$-dense set in $X$ and $\{3\}$ is not a $w^2w^1$-dense set in $X$.

Theorem 3.3.3. Let $(X, w^1, w^2)$ be a bi-w space and $A$ be a subset of $X$. If $A$ is a $w^iw^j$-dense set in $X$, then for any nonempty bi-w-closed subset $F$ of $X$ such that $A \subseteq F$, we have $F = X$ .

Proof. Suppose that $A$ is a $w^iw^j$-dense set in $X$ and $F$ is a bi-w-closed subset of $X$ such that $A \subseteq F$. Since $A$ is a $w^iw^j$-dense set in $X$, $X = c_{w^i}(c_{w^j}(A))$. By assumption, $F$ is bi-w-closed and $A \subseteq F$, it follows that $X = c_{w^i}(c_{w^j}(A)) \subseteq c_{w^i}(c_{w^j}(F)) = F$ . Hence $F = X$.

Remark By Theorem 3.3.3, if $A$ is a $w^iw^j$-dense set in $X$, then only $X$ is bi-w-closed set in $X$ containing $A$. Moreover, it is not true if $F$ is not bi-w-closed. We can be seen from the following example.

Example 3.3.4. Let $X = \{1,2,3\}$. Define weak structures $w^1$ and $w^2$ on $X$ as follows: $w^1 = \{\emptyset, \{1\}, \{1,3\}\}$ and $w^2 = \{\emptyset, \{1\}, \{2\}, \{1,3\}\}$. Then $c_{w^1}(c_{w^2}(\{1\})) = X$. Hence $\{1\}$ is a $w^1w^2$-dense set in $X$, but $\{1\}$ is not a $w^1w^2$-closed set in $X$.

Theorem 3.3.5. Let $(X, w^1, w^2)$ be a bi-w space and $A$ be a subset of $X$. The following are equivalent.

1. If $F$ is a nonempty bi-w-closed subset of $X$ such that $A \subseteq F$, then $F = X$.
2. $G \cap A \neq \emptyset$ for any nonempty bi-w-open subset $G$ of $X$.

Proof. (1. ⇒ 2.) Assume that if $F$ is a nonempty bi-w-closed subset of $X$ such that $A \subseteq F$, then $F = X$. Suppose that $G \cap A = \emptyset$ for some nonempty bi-w-open subset $G$ of $X$. Thus $A \subseteq X\backslash G$. Since $G$ is bi-w-open, $X\backslash G$ is bi-w-closed. By assumption, we have $X\backslash G = X$. Therefore $G = \emptyset$, this is contradiction. Hence $G \cap A \neq \emptyset$ for any nonempty bi-w-open subset $G$ of $X$.

(2. ⇒ 1.) Assume that 2. holds, and $F$ is a nonempty bi-w-closed subset of $X$ such that $A \subseteq F$. Suppose that $F \neq X$. Thus $X\backslash F$ is a nonempty bi-w-open subset of $X$. By assumption, we have $(X\backslash F) \cap A \neq \emptyset$. This is contradiction with $A \subseteq F$. Therefore $F = X$.

Corollary 3.3.6. Let $(X, w^1, w^2)$ be a bi-w space and $A \subseteq X$. If $A$ is a $w^i w^j$-dense set in $X$, then $G \cap A \neq \emptyset$ for any nonempty bi-w-open subset $G$ of $X$.

Proof. It follows from Theorem 3.3.3 and Theorem 3.3.5.

Theorem 3.3.7. Let $(X, w^1, w^2)$ be a bi-w space and $A$ be a subset of $X$. Then $i_{w^i}(i_{w^j}(X \backslash A)) = \emptyset$ if and only if $A$ is a $w^i w^j$-dense set in $X$.

Proof. ($\Rightarrow$) Assume that $i_{w^i}(i_{w^j}(X \backslash A)) = \emptyset$. Thus $X \backslash c_{w^i}(c_{w^j}(A)) = \emptyset$, it follows that $c_{w^i}(c_{w^j}(A)) = X$. Therefore $A$ is a $w^i w^j$-dense set in $X$.

($\Leftarrow$) Suppose that $A$ is a $w^i w^j$-dense set in $X$. Then we have $c_{w^i}(c_{w^j}(A)) = X$, and so $i_{w^i}(i_{w^j}(X \backslash A)) = X \backslash c_{w^i}(c_{w^j}(A)) = \emptyset$.

Theorem 3.3.8. Let $(X, w^1, w^2)$ be a bi-w space and $A$ be a subset of $X$. Then $A$ is a $w^i w^j$-dense set in $X$ if and only if $wExt_{ij}(A) = \emptyset$.

Proof. ($\Rightarrow$) Suppose that $A$ is a $w^i w^j$-dense set in $X$. Then $wExt_{ij}(A) = X \backslash c_{w^i}(c_{w^j}(A)) = X \backslash X = \emptyset$.

($\Leftarrow$) Assume that $wExt_{ij}(A) = \emptyset$. Then $X \backslash c_{w^i}(c_{w^j}(A)) = \emptyset$. It follows that $c_{w^i}(c_{w^j}(A)) = X$. Therefore $A$ is a $w^i w^j$-dense set in $X$.

## Discussion

In 2011 and 2012, the notions and properties of boundary sets, exterior sets and dense sets in bi-minimal structure spaces are studied Sompong. In 2017, Puiwong et al. studied bi-weak structure spaces. It is obvious that a bi-minimal structure space is a bi-weak structure space. We investigated the above concepts into bi-weak structure spaces. The similar properties in bi-minimal structure spaces are obtained in bi-weak structure spaces except the property of exterior of empty set, that is, the exterior of empty set is not empty set in bi-weak structure spaces.

## Conclusions

In this paper, we introduced and studied boundary sets, exterior sets and dense sets in a bi-weak structure space. We obtained some of their properties. In particular, we gave some characterizations of closed sets in a bi-weak structure space. That is, $A$ is bi-w-closed if and only if $wBdr_{ij}(A) \subseteq A$. And $A$ is bi-w-closed if and only if $wExt_{ij}(A) = X \backslash A$. Moreover, we also obtained a characterization of dense sets, i.e., $A$ is a $w^i w^j$-dense set in $X$ if and only if $wExt_{ij}(A) = \emptyset$.

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