การพิสูจน์ของกึ่งสาทิสสัณฐานแบบอื่น

Another Proof of Half Homomorphisms

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บทคัดย่อ

สกอตได้พิสูจน์ใน (Scott, 1957) ว่ากึ่งสาทิสสัณฐานของกรุปจะเป็นสาทิสสัณฐานหรือปฏิสาทิสสัณฐานเท่านั้น บทพิสูจน์ของสกอตใช้สมบัติของกึ่งสมสัณฐานของกึ่งกรุปที่มีสมบัติการตัดออก ในอีกมุมมองหนึ่งแมนส์ฟิลด์ได้พิสูจน์ไว้ใน (Mansfield, 1992) ว่าดีเทอร์มิแนนต์ของกรุปสามารถระบุกรุปตั้งต้นได้ เราจะพิสูจน์ในอีกวิธีหนึ่งว่ากึ่งสาทิสสัณฐานของกรุปจะเป็นสาทิสสัณฐานหรือปฏิสาทิสสัณฐานเท่านั้น โดยใช้กระบวนการของแมนส์ฟิลด์ในการระบุกรุปตั้งต้นของดีเทอร์มิแนนต์ของกรุป

คำสำคัญ : สาทิสสัณฐาน, ปฏิสาทิสสัณฐาน, ดีเทอร์มิแนนต์ของกรุป

Abstract

Scott proves in (Scott, 1957) that a half-homomorphism of a group is either a homomorphism or an anti-homomorphism. The proof given by Scott relies on properties of half-isomorphisms of a cancellation semi-group. In a different point of view, Mansfield proves that, in (Mansfield, 1992), a group determinant determines the underlying group. We give an alternate proof that a half-homomorphism of a group is either a homomorphism or an anti-homomorphism by using Manfields's process to determine the underlying group of a group determinant.

Keywords : half-homomorphism, anti-homomorphism, group determinant

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Introduction

Given groups G, H, we say that a function $\phi : G \to H$ is a semi-homomorphism if

$$\phi(ghg) = \phi(g)\phi(h)\phi(g) \tag{1}$$

for all $g, h \in G$. Semi-isomorphisms and semi-automorphisms can be defined in the same manner. Two obvious examples of semi-homomorphisms are homomorphisms and anti-homomorphisms. A homomorphism is a map $\phi : G \to H$ such that $\phi(gh) = \phi(g)\phi(h)$ for all $g, h \in G$, while an anti-homomorphism is a map $\phi : G \to H$ satisfying $\phi(gh) = \phi(h)\phi(g)$ for all $g, h \in G$. Simply speaking, an anti-homomorphism is a homomorphism that reverse the order of multiplication.

The study of semi-isomorphism has been established, most regarding when a semi-automorphism of a group is either an automorphism or anti-automorphism. We say that such semi-automorphisms are disjunctive. For example, let X be a set and $S(X)$, $A(X)$ be the symmetric group and the alternating group on elements of X. The author of (Scott, 1969) shows that every semi-automorphism of any subgroup $G \subseteq S(X)$ containing $A(X)$ comes from inner automorphisms and function $x \mapsto x^{-1}$. As in (Sullivan, 1983), every semi-automorphism of a non-abelian simple group with an element of order 2 is disjunctive. A good summary of results on semi-automorphism of various algebraic structures can be found in (Sullivan, 1983) and (Suliivan, 1985). Note from (Dinkines, 1951), it is evident that one can construct a semi-automorphism on a direct product of non-abelian groups to be automorphism on one factor and anti-automorphism on the other. As such, there are non-disjunctive semi-automorphisms.

Methods

It is known that by adding an extra property to semi-automorphisms, it will be disjunctive.

Definition 1. Let G, H be groups. A map $\phi : G \to H$ is a half-homomorphism if $\phi(gh) = \phi(g)\phi(h)$ or $\phi(h)\phi(g)$ for all $g, h \in G$.

Half-automorphisms are likewise defined. It is proved in (Scott, 1957) that

Theorem 2. A half-homomorphism is either homomorphism or anti-homomorphism.

In this article we provide another proof of the same result inspired from a different point of view.

Defnition 3. Let G be a finite abelian group of cardinality n, and let $\left\{x_g\right\}$ be commuting indeterminates indexed by elements of G. The group matrix is an $n \times n$ matrix given by $\left\{x_{gh^{-1}}\right\}$ Its determinant Θ_G is called the group determinant associated with G.

Once expanded, Θ_G is then a homogeneous polynomial of degree *n* with x_g's as its variables. The first study of Θ_G is done by Dedekind and Frobenius on the factorization of Θ_G over $\mathbb{C}$, which lays a foundation of today's representation theory, see (Lam, 1998).

The group determinant compress all information of *G* into a single polynomial. In (Mansfield, 1992), Mansfield shows that every group determinant determines its underlying group. Inspired by the proof of Lemma 4 in (Mansfield, 1992) to determine the multiplication in G after knowing Θ_G we produce another proof of Theorem 2.

The proof of Theorem 2 given by Scott relies on properties of half-isomorphisms of a cancellation semi-group to determine the order of multiplication of a half-homomorphism. We will prove in the next section that the steps Mansfield uses to determine the group structure from a group determinant can also determine the order of multiplication of a half-homomorphism.

Results

We give another proof of Theorem 2 as follows:

Proof. First we will show that it suffices to prove that every half-isomorphism is either an isomorphism or an anti-isomorphism. Let $\phi : G \rightarrow H$ be a half-homomorphism. It is clear that its Kernel $N = Ker(\phi)$ is a normal subgroup of *G*. The induced map from G/N to $\phi(g)$ by $gN \mapsto \phi(g)$ is then a half-isomorphism.

It also suffices to assume that *G* is non-abelian, as every half-homomorphism is a homomorphism in abelian groups. Thus there exist $g, h \in G$ such that $gh \neq hg$. For simplicity, we will write g' for its image $\phi(g)$.

Suppose now that $\phi(gh) = \phi(g)\phi(h)$. We claim that $\phi(rs) = \phi(r)\phi(s)$ for any $r, s \in G$. We break down the proof of the claim into the following steps,

(i) If $(rs)' = r's'$, then $(sr)' = s'r'$,

(ii) $(ghr)' = g'h'r'$,

(iii) $(gr)' = g'r'$, $(hr)' = h'r'$,

(iv) $(rs)' = r's'$.

The idea is to consider all multiplication orders and show that there is only way to do it without reaching a contradiction.

Proof of (i). Since ϕ is a half-isomorphism, $(sr)' = r's'$ or $s'r'$. But $(rs)' = r's'$ and ϕ is a bijection, then $s'r'$ is the only possible value for $(sr)'$.

Proof of (ii). We will compute possible values for $(ghr)'$ in two different ways. Since multiplication is associative, it is clear that $((gh)r)' = (g(hr))'$. Recalling that $(gh)' = g'h'$, we must have

$$((gh)r)' \in \{g'h'r', r'g'h'\} \text{ and} \tag{2}$$

$$(g(hr))' \in \{g'h'r', g'r'h', h'r'g', r'h'g'\}. \tag{3}$$

If $(ghr)' = g'h'r'$, there is nothing to prove. So suppose $(ghr)' \neq g'h'r'$. Then Equation 2 implies that $(ghr)' = r'g'h'$, It follows from Equation 3 that

$$g'h'r' \neq r'g'h' \in \{g'r'h', h'r'g', r'h'g'\}. \tag{4}$$

We will show that is not possible.

Case 1, $(ghr)' = r'g'h' = g'r'h'$. Then $r'g' = g'r'$. Since ϕ is a half-isomorphism, then $(rg)' = r'g'$ Together with the fact that $(gh)' = g'h'$, one gets

$$((rg)h)' \in \{r'g'h', h'r'g'\} \text{ and} \tag{5}$$

$$(r(gh))' \in \{r'g'h', g'h'r'\}. \tag{6}$$

If $(rgh)' \neq r'g'h'$, then Equation 5 and 6 implies that

$$h'r'g' = h'g'r' = g'h'r' \tag{7}$$

$$h'g' = g'h'. \tag{8}$$

contradicting that *g, h* do not commute. Hence $(r(gh))' = r'(g'h')$. Using part (1) to reverse the order of multiplication to get $((gh)r)' = g'h'r'$, which contradicts the assumption that they are not equal. Thus this case can be eliminated.

Case 2, $(ghr)' = r'g'h' = h'r'g'$. It is a simple observation that $(h')^{-1} = (h^{-1})'$, which makes

$$(h^{-1})'(ghr)' = (ghr)'(h^{-1})' = r'g'. \tag{9}$$

Taking the preimage and applying part (i), we get

$$h^{-1}ghr = ghrh^{-1} = rg \text{ or } gr. \tag{10}$$

If $gr = h^{-1}ghr$, then $gh = hg$, making *g, h* commutative. Therefore, from Equation 10,

$$ghrh^{-1} = rg \tag{11}$$

$$ghr = rgh \tag{12}$$

$$((gh)r)' = (r(gh))'. \tag{13}$$

We recall that $(gh)' = g'h'$. Again by using part (1), we see that $r'g'h' = g'h'r'$. But this contradicts Equation 4.

Case 3, $r'g'h' = r'h'g'$. This case simply does not occur as $gh \neq hg$. We just finished the proof of part (ii).

Proof of (iii). The proof of $(gr)' = g'r'$ is achieved by replacing *g, h* in part (ii) by gh and h^{-1} respectively.

Similarly assigning $g \mapsto g^{-1}$, $h \mapsto gh$, one can easily proves that $(hr)' = h'r'$.

Proof of (iv). We consider two cases. One is when *r* or *s* does not commute with either *g* or *h*, and the other when both *r, s* commute with *g, h*.

First case. Without loss of generality suppose *r* does not compute with *g*. Then we can assign $h \mapsto r$, $r \mapsto s$ and apply part (iii) to establish that $(rs)' = r's'$.

Finally, we consider the second case, when *r, s* commute with both *g, h*. Now we will compute all possible values for $(ghrs)'$.

$$((gr)(hs))' \in \{ (gr)'(hs)', (hs)'(gr)' \} = \{ g'h'r's', h'g's'r' \}$$

$$((gh)(rs))' \in \{ g'h'r's', g'h's'r' \}. \tag{14}$$

If $(ghrs)' \neq g'h'r's'$, then Equation 14 forces

$$h'g's'r' = g'h's'r' \tag{15}$$

$$h'g' = g'h'. \tag{16}$$

which is not possible. Hence we finally arrive at the conclusion

$$((gh)(rs))' = g'h'r's'. \tag{17}$$

which yields $(rs)' = r's'$. We have just shown that if $\phi(gh) = \phi(g)\phi(h)$ then ϕ must be an isomorphism. In the same manner it is clear that if $\phi(gh) = \phi(h)\phi(g)$, then ϕ must be an anti-isomorphism.

Discussion

We end this article with a short discussion about automorphisms and anti-automorphisms of a group G, denoted by $\mathrm{Aut}(G)$ and $\mathrm{Anti}(G)$ respectively. When G is abelian $\mathrm{Aut}(G) = \mathrm{Anti}(G)$, and they form a group of automorphisms. When G is non-abelian, however, $\mathrm{Anti}(G)$ is not a group as it has no identity. In this case, $\mathrm{Aut}(G) \cap \mathrm{Anti}(G)$ is an empty set as any element in $\mathrm{Anti}(G)$ always reverses the order of multiplication of non-commutative elements in G. Note that composition by the inversion map $x \mapsto x^{-1}$ induces a bijection between $\mathrm{Aut}(G)$ and $\mathrm{Anti}(G)$ Since inversion map is an anti-automorphism, it is clear that $\mathrm{Aut}(G) \subseteq \langle \mathrm{Anti}(G) \rangle$.
Now we will explicitly describe $\langle \mathrm{Anti}(G) \rangle$.

Lemma 4. $\langle \mathrm{Anti}(G) \rangle = \mathrm{Aut}(G) \cup \mathrm{Anti}(G)$.

Proof. The fact is obvious when G is abelian, so assume that G is non-abelian. Earlier we show that $\langle \mathrm{Anti}(G) \rangle \supseteq \mathrm{Aut}(G) \cup \mathrm{Anti}(G)$. A straightforward computation also shows that for any $\alpha, \beta \in \mathrm{Anti}(G)$, $\gamma \in \mathrm{Aut}(G)$, $\alpha \circ \beta \in \mathrm{Aut}(G)$ and $\alpha \circ \gamma \in \mathrm{Anti}(G)$, as α, β reverse the order of multiplication. Therefore, any element generated by $\mathrm{Anti}(G)$ is contained in $\mathrm{Aut}(G) \cup \mathrm{Anti}(G)$.

We end the article with the following corollary:

Corollary 5. The set of all half-automorphisms on a group G forms a group generated by anti-automorphisms of G.

Conclusion

The group determinant compress the structure of an entire group into a single polynomial. The process to determine the multiplicative order of the group operation limits a half-homomorphism of a group to be either a homomorphism or an anti-homomorphism. As a result, the set of half-homomorphisms forms a group generated by anti-homomorphisms.

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