วารสารวิทยาศาสตร์บูรพา ปีที่ 28 (ฉบับที่ 3) กันยายน – ธันวาคม พ.ศ. 2566
BURAPHA SCIENCE JOURNAL Volume 28 (No.3) September – December 2023 บทความวิจัย

การดำเนินการทวิภาคที่เปลี่ยนหมู่ของฟูลเทอมที่อันดับลด

Associative Binary Operations of Order-Decreasing Full Terms

ทศพร คำดวง*

Thodsaporn Kumduang*

สาขาวิชาคณิตศาสตร์ คณะวิทยาศาสตร์และเทคโนโลยี มหาวิทยาลัยเทคโนโลยีราชมงคลรัตนโกสินทร์ ประเทศไทย

Department of Mathematics, Faculty of Science and Technology, Rajamangala University of Technology Rattanakosin, Thailand

Received : 17 October 2022

Revised : 25 April 2023

Accepted : 22 May 2023

บทคัดย่อ

เซตของฟูลเทอมที่อันดับลด $W_{\tau_n}^{OD_n}(X_n)$ เป็นเซตของเทอมชนิดพิเศษที่สร้างมาจากการแปลงเต็มที่อันดับลดบนโซ่จำกัด $\{1,\ldots,n\}$ และตัวแปรจากพยัญชนะ X_n เซตของฟูลเทอมที่อันดับลดทั้งหมดถูกทำให้มีสมบัติปิดภายใต้การดำเนินการซุปเปอร์โพสิชันที่ซึ่งสอดคล้องกับกฎซุปเปอร์ของการเปลี่ยนหมู่ ในงานวิจัยนี้การดำเนินการทวิภาค $+,*,\cdot_{x_i}$ บนเซตของฟูลเทอมที่อันดับลดซึ่งก่อกำเนิดจากซุปเปอร์โพสิชันได้ถูกนิยามและความสอดคล้องกับสมบัติการเปลี่ยนหมู่ได้รับการพิสูจน์ นอกจกานั้นเราพิจารณาภาษาต้นไม้ของฟูลเทอมที่อันดับลดและนิยามการดำเนินการของภาษาต้นไม้ดังกล่าว ทฤษฎีบทการฝังของกึ่งกรุปของฟูลเทอมที่อันดับลดไปในกึ่งกรุปของภาษาต้นไม้ที่สร้างมาจากฟูลเทอมที่อันดับลดได้รับการนำเสนอ

คำสำคัญ : ฟูลเทอมที่อันดับลด ; การแปลงเต็มที่อันดับลด ; การเปลี่ยนหมู่ ; การดำเนินการทวิภาค

Abstract

Order-decreasing full terms are special types of terms derived from order-decreasing full transformations on a finite chain $\{1,\ldots,n\}$ and variables from an alphabet X_n. The set of all order-decreasing full terms is closed under the superposition operation such that the superassociative law holds. In this work, three binary operations on the set of all order-decreasing full terms of type τ_n induced by the superposition are defined and the satisfaction with associative property is proved. We also consider tree languages of order-decreasing full terms and define their operation. Embedding theorems of semigroups of order-decreasing full terms into semigroups of tree languages constructed from order-decreasing full terms are provided.

Keywords : order-decreasing full term; order-decreasing full transformation; associativity; binary operation

*Corresponding author. E-mail : thodsaporn.kum@rmutr.ac.th

Introduction

It is widely accepted that the concept of terms is one of the most important ideas in the study of algebra, especially applications of algebra in theoretical computer science. Recall from (Wattanatripop & Changphas, 2020) that terms are formal expressions induced by variables from an alphabet and compositions of operation symbols. By the definition, an n-ary term of type τ is inductively defined by the following steps: Every variable x_i in $X_n := \{x_1,...,x_n\}$ is an n-ary term of type τ and $f_i(t_1,...,t_{n_i})$ is also an n-ary term of type τ if $t_1,...,t_{n_i}$ are already known. There are many ways to define several classes of terms, for example, linear terms (Denecke, 2016) and terms with fixed variables (Wattanatripop & Changphas, 2020).

In this work, we focus on full terms introduced by K. Denecke and his colleagues in (Denecke, 2016) and (Wattanatripop & Changphas 2019). Let τ_n be a type of all arity n for all $i \in I$, i.e., $\tau_n = (n_i)_{i\in I}$ and $n_i = n$ for all $i \in I$. By the symbol T_n, we denote the set of all mappings from $\{1,...,n\}$ to itself. It was mentioned in (Umar, 1992) that the set T_n together with a binary composition of functions forms a semigroup called a transformation semigroup. For any mapping α in T_n, an n-ary full term of type τ_n is defined by the following steps:

1. $f_i(x_{\alpha(1)},...,x_{\alpha(n)})$ is an n-ary full term of type τ_n where $\alpha \in T_n$,
2. if $t_1,...,t_n$ are n-ary full terms of type τ_n, then $f_i(t_1,...,t_n)$ is an n-ary full term of type τ_n.

The set of all n-ary full terms of type τ_n is denoted by $W_{\tau_n}{}^F(X_n)$. Let us consider the following example. For a type $\tau_3 = (3,3)$ with two ternary operation symbols f and g, we have

$$f(x_1,x_2,x_3),\ g(x_1,x_1,x_1),\ f(x_2,x_1,x_3),\ f(x_3,x_3,x_2),\ g(x_2,x_1,x_1),$$
$$f(g(x_1,x_2,x_3), f(x_3,x_1,x_2), f(x_2,x_1,x_3)),\ g(f(x_2,x_3,x_3), f(x_3,x_2,x_1), g(x_2,x_2,x_3))$$

are examples of full terms in $W_{(3,3)}{}^F(X_3)$.

One of the generalizations of terms in the study of universal algebra and automata theory is a set of terms. In fact, we call sets of terms tree languages. The set of all subsets or tree languages of all n-ary full terms of type τ_n is denoted by $P(W_{\tau_n}{}^F(X_n))$. For example, we have

$$\{f(x_1,x_2,x_3)\},\ \{g(x_1,x_1,x_1), f(x_2,x_2,x_3)\}, \{f(g(x_1,x_2,x_3), f(x_3,x_1,x_2), f(x_2,x_1,x_3))\},$$
$$\{f(x_3,x_3,x_3), g(x_1,x_1,x_1),\ f(f(x_2,x_3,x_3), g(x_3,x_2,x_1), g(x_2,x_2,x_3))\}$$

are examples of tree languages in $P(W_{(3,3)}{}^F(X_3))$. However, $\{f(x_1,x_2,f(x_2,x_3,x_3))\} \notin P(W_{(3,3)}{}^F(X_3))$. To compute the result of tree languages of full terms, in (Wattanatripop & Changphas, 2021), a non-deterministic superposition operation on the set $P(W_{\tau_n}{}^F(X_n))$ was firstly defined. By the definition, a mapping

$$\hat{S}^n: P\left(W^F_{\tau_n}(X_n)\right)^{n+1} \to P\left(W^F_{\tau_n}(X_n)\right)$$

is defined as follows:

1. $\hat{S}^n\left(\{f_i\left(x_{\alpha(1)}, \ldots, x_{\alpha(n)}\right)\}, B_1, \ldots, B_n\right) = \left\{f_i\left(r_{\alpha(1)}, \ldots, r_{\alpha(n)}\right) \middle| r_{\alpha(j)} \in B_{\alpha(j)}, j = 1, \ldots, n\right\}$,
2. $\hat{S}^n(\{f_i(t_1, \ldots, t_n)\}, B_1, \ldots, B_n) = \left\{f_i(r_1, \ldots, r_n) \middle| r_j \in \hat{S}^n\left(\{t_j\}, B_1, \ldots, B_n\right), j = 1, \ldots, n\right\}$,
3. $\hat{S}^n(A, B_1, \ldots, B_n) = \bigcup_{a\in A}\{\hat{S}^n(\{a\}, B_1, \ldots, B_n)\}$ if $|A| > 1$,
4. $\hat{S}^n(A, B_1, \ldots, B_n) = \emptyset$ if $A = \emptyset$ or $B_j = \emptyset$ for some j.

As a consequence, the Menger algebra $(P\left(W^F_{\tau_n}(X_n)\right), \hat{S}^n)$ of type $(n+1)$ is obtained.

In 2021, the concept of order-decreasing full terms was introduced by K. Wattanatripop and T. Changphas (Wattanatripop & Changphas, 2021). We now recall the definition of a semigroup of order-decreasing transformations (Sun, 2020). The set

$$OD_n = \{\alpha \in T_n \mid \alpha(x) \le x, \forall x \in \{1, \ldots, n\}\}$$

whose elements are called order-decreasing transformations equipped with the usual composition of functions is a semigroup. For more details, we refer to (Yang & Yang, 2012). Applying this structure, a particular class of full terms was given. Actually, an n-ary order-decreasing full term of type τ_n is inductively defined in the following setting:

1. $f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$ is an n-ary full term of type τ_n where $\alpha \in OD_n$,
2. if $t_1, \ldots, t_n$ are n-ary full terms of type τ_n, then $f_i(t_1, \ldots, t_n)$ is an n-ary full term of type τ_n.

Let $W_{\tau_n}^{OD_n}(X_n)$ be the set of all n-ary order-decreasing full terms of type τ_n.

For example, let $\tau_3 = (3,3,3)$ be a type with three ternary operation symbols Δ, Θ, Ω. Then we have

$$\Delta(x_1, x_2, x_3),\ \Theta(x_1, x_2, x_1),\ \Omega(x_1, x_1, x_3) \in W_{(3,3,3)}^{OD_3}(X_3)$$

because $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix} \in OD_3$. On the other hand, we obtain

$$\Delta(x_2, x_3, x_1),\ \Theta(x_1, x_3, x_1),\ \Omega(x_1, x_3, x_2) \notin W_{(3,3,3)}^{OD_3}(X_3)$$

because $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \notin OD_3$.

The superposition operation S^n on the set $W_{\tau_n}^{OD_n}(X_n)$ of all n-ary order-decreasing full terms of type τ_n was firstly defined in the paper (Wattanatripop & Changphas, 2021). In fact, it is a mapping

$$S^n : W_{\tau_n}^{OD_n}(X_n)^{n+1} \to W_{\tau_n}^{OD_n}(X_n)$$

defined by

1. $S^n(f_i(x_{\alpha(1)},...,x_{\alpha(n)}),s_1,...,s_n)=f_i(s_{\alpha(1)},...,s_{\alpha(n)})$,
2. $S^n(f_i(t_1,...,t_n),s_1,...,s_n)=f_i(S^n(t_1,s_1,...,s_n),...,S^n(t_n,s_1,...,s_n))$.

As a result, the algebra

$$MA_{OD_n}(\tau_n):=(W_{\tau_n}{}^{OD_n}(X_n),S^n)$$

consisting of the set of all n-ary order-decreasing full terms of type τ_n and the superposition S^n of type $(n+1)$ is constructed. It is not hard to verify that the operation S^n defined on $MA_{OD_n}(\tau_n)$ satisfies the following identity:

$$S^n(S^n(t,s_1,...,s_n),u_1,...,u_n)=S^n(t,S^n(s_1,u_1,...,u_n),...,S^n(s_n,u_1,...,u_n)).$$

This equation also known as the superassociative law which always plays a key role in the theory of multiplace functions and superassociative algebras. For more details, we refer to (Chansuriya, 2021, (Kumduang & Leeratanavalee, 2021), (Dudek & Trokhimenko, 2018, 2021), (Kumduang & Sriwongsa, 2022) and (Phuapong & Pookpienlert, 2022).

In this work, we aim to define binary operations on the $W_{\tau_n}{}^{OD_n}(X_n)$ of all order-decreasing full terms derived from the superposition S^n and prove that these operations are associative. For tree languages of order-decreasing full terms, binary operations induced by a non-deterministic operation $\hat{S}^n$ are introduced and the fact that these operations satisfy an associativity is given. Finally, this work is devoted to the embeddability of semigroups of order-decreasing full terms into semigroups of tree languages of order-decreasing full terms.

Methods

We begin this section with giving three kinds of binary operations defined on the set of all order-decreasing full terms.

For any s,t in $W_{\tau_n}{}^{OD_n}(X_n)$, we define the binary operation

$$+:W_{\tau_n}{}^{OD_n}(X_n)\times W_{\tau_n}{}^{OD_n}(X_n)\to W_{\tau_n}{}^{OD_n}(X_n)$$

by

$$s+t=S^n(s,\underbrace{t,...,t}_{n}).$$

Then we have:

Theorem 1 $\left(W_{\tau_n}{}^{OD_n}(X_n),+\right)$ is a semigroup.

Proof We show that the binary operation $+$ is associative. For this, let s,t,u be arbitrary elements in $W_{\tau_n}^{OD_n}(X_n)$. By the superassociativity of the operation S^n, we have

$$\begin{aligned}(s+t)+u &= S^n(s,t,...,t)+u = S^n(S^n(s,t,...,t),u,...,u)\\ &= S^n(s,S^n(t,u,...,u),...,S^n(t,u,...,u)) = s+S^n(t,u,...,u) = s+(t+u).\end{aligned}$$

Therefore, $\left(W_{\tau_n}^{OD_n}(X_n),+\right)$ is a semigroup.

The obtained semigroup in Theorem 1 is called the *diagonal semigroup* derived from the Menger algebra $MA_{OD_n}(\tau_n)$.

Let s and t be order-decreasing full terms in the set $W_{\tau_n}^{OD_n}(X_n)$. For each $i=1,...,n$, the binary operation

$$\cdot_{x_i} : W_{\tau_n}^{OD_n}(X_n)\times W_{\tau_n}^{OD_n}(X_n)\to W_{\tau_n}^{OD_n}(X_n)$$

can be defined by

$$s\cdot_{x_i} t = S^n(s,x_1,...,x_{i-1},t,x_{i+1},...,x_n).$$

The fact that the binary operation $\cdot_{x_i}$ satisfies the associative law is now proved.

Theorem 2 $\left(W_{\tau_n}^{OD_n}(X_n),\cdot_{x_i}\right)$ is a semigroup.

Proof To show that the binary operation $\cdot_{x_i}$ is associative, let s,t,u be arbitrary full terms in $W_{\tau_n}^{OD_n}(X_n)$. Due to the satisfaction of S^n of the superassociative law over the set $W_{\tau_n}^{OD_n}(X_n)$, we obtain

$$\begin{aligned}(s\cdot_{x_i} t)\cdot_{x_i} u &= S^n(s,x_1,...,x_{i-1},t,x_{i+1},...,x_n)\cdot_{x_i} u\\ &= S^n(S^n(s,x_1,...,x_{i-1},t,x_{i+1},...,x_n),x_1,...,x_{i-1},u,x_{i+1},...,x_n)\\ &= S^n(s,x_1,...,x_{i-1},S^n(t,x_1,...,x_{i-1},u,x_{i+1},...,x_n),x_{i+1},...,x_n)\\ &= s\cdot_{x_i} S^n(t,x_1,...,x_{i-1},u,x_{i+1},...,x_n)\\ &= s\cdot_{x_i}(t\cdot_{x_i} u),\end{aligned}$$

which shows that $\left(W_{\tau_n}^{OD_n}(X_n),\cdot_{x_i}\right)$ forms a semigroup.

On the Cartesian product $W_{\tau_n}^{OD_n}(X_n)^n$ of n-tuples of order-decreasing full terms of type τ_n, we define the binary operation

$$*: W_{\tau_n}^{OD_n}(X_n)^n\times W_{\tau_n}^{OD_n}(X_n)^n\to W_{\tau_n}^{OD_n}(X_n)^n$$

by

$$(s_1,....,s_n)*(t_1,....,t_n) = (S^n(s_1,t_1,....,t_n),...,S^n(s_n,t_1,....,t_n))$$

for all $(s_1,....,s_n),(t_1,....,t_n)\in W_{\tau_n}^{OD_n}(X_n)^n$.

The following theorem shows that the operation $*$ is associative and forms a semigroup.

Theorem 3 $\left(W_{\tau_n}{}^{OD_n}(X_n)^n,*\right)$ is a semigroup.

Proof Let $(s_1,....,s_n),(t_1,....,t_n),(u_1,....,u_n)\in W_{\tau_n}{}^{OD_n}(X_n)^n$. Then we have

$$((s_1,....,s_n)*(t_1,....,t_n))*(u_1,....,u_n)$$
$$=(S^n(s_1,t_1,....,t_n),...,S^n(s_n,t_1,....,t_n))*(u_1,....,u_n)$$
$$=(S^n(S^n(s_1,t_1,....,t_n),u_1,....,u_n),...,S^n(S^n(s_n,t_1,....,t_n),u_1,....,u_n))$$
$$=(S^n(s_1,S^n(t_1,u_1,....,u_n),...,S^n(t_n,u_1,....,u_n)),...,S^n(s_n,S^n(t_1,u_1,....,u_n),...,S^n(t_n,u_1,....,u_n)))$$
$$=(s_1,....,s_n)*(S^n(t_1,u_1,...,u_n),...,S^n(t_n,u_1,...,u_n))$$
$$=(s_1,....,s_n)*((t_1,...,t_n)*(u_1,...,u_n)).$$

Consequently, $\left(W_{\tau_n}{}^{OD_n}(X_n)^n,*\right)$ is a semigroup.

We call the obtained semigroup in Theorem 3 the *binary comitant* of the Menger algebra $MA_{OD_n}(\tau_n)$.

On the set $W_{\tau_n}{}^{OD_n}(X_n)$, one can consider its power set, i.e., $P(W_{\tau_n}{}^{OD_n}(X_n))$. Each element in $P(W_{\tau_n}{}^{OD_n}(X_n))$ is called a *tree language of order-decreasing full terms*. Let see some examples. Consider a type $\tau_4=(4)$ with one quaternary operation symbol ∇. It is clear that

$$\varnothing,\{\nabla(x_1,x_2,x_3,x_4)\},\{\nabla(x_2,x_4,x_4,x_2)\},\{\nabla(x_2,x_1,x_4,x_3),\nabla(x_3,x_1,x_1,x_1)\}$$

are examples of tree languages in $P(W_{(4)}{}^{OD_4}(X_4))$.

Normally, by a subalgebra of a Menger algebra (G,o), we mean a nonempty set H of G which is closed under the operation o defined on G. So we prove the following theorem.

Theorem 4 $(P\left(W_{\tau_n}^{OD_n}(X_n)\right),\hat{S}^n)$ is a subalgebra of the Menger algebra $(P\left(W_{\tau_n}^{F}(X_n)\right),\hat{S}^n)$.

Proof Obviously, $P(W_{\tau_n}{}^{OD_n}(X_n))\subseteq P(W_{\tau_n}{}^{F}(X_n))$. Now we let $A,B_1,...,B_n$ be tree languages of order-decreasing full terms. We give the proof by a structure of a set A. If A or $B_j=\varnothing$ for some $j=1,...,n,$ $\hat{S}^n(A,B_1,\dots,B_n)=\emptyset$, which implies that $\hat{S}^n(A,B_1,\dots,B_n)\in P\left(W_{\tau_n}^{OD_n}(X_n)\right)$. If A is a singleton set, we consider in two cases. If $A=\{f_i(x_{\alpha(1)},...,x_{\alpha(n)})\}$ where α is an order-decreasing full term on $\{1,...,n\},$ then we have

$$\hat{S}^n(\{f_i(x_{\alpha(1)},\dots,x_{\alpha(n)})\},B_1,\dots,B_n)=\{f_i(r_{\alpha(1)},\dots,r_{\alpha(n)})|r_{\alpha(j)}\in B_{\alpha(j)},j=1,\dots,n\}\in P\left(W_{\tau_n}^{OD_n}(X_n)\right)$$

because $B_{\alpha(j)}\in P(W_{\tau_n}{}^{OD_n}(X_n))$ for all $j=1,...,n$. Assume that $A=\{f_i(t_1,...,t_n)\}$ and that each $\hat{S}^n(\{t_j\},B_1,\dots,B_n)$ is an order-decreasing full term on $\{1,...,n\}$. It follows that

$$\hat{S}^n(A,B_1,\dots,B_n)=\hat{S}^n(\{f_i(t_1,\dots,t_n)\},B_1,\dots,B_n)$$
$$=\{f_i(r_{\alpha(1)},\dots,r_{\alpha(n)})|r_{\alpha(j)}\in\hat{S}^n(\{t_j\},B_1,\dots,B_n),j=1,\dots,n\}$$

belongs to the set $P(W_{\tau_n}^{OD_n}(X_n))$. In the case when a set A is a nonempty arbitrary set, we obtain that $\hat{S}^n(A, B_1, \dots, B_n) = \bigcup_{a \in A}\{\hat{S}^n(\{a\}, B_1, \dots, B_n)\}$ is in $P(W_{\tau_n}^{OD_n}(X_n))$ because $\hat{S}^n(\{a\}, B_1, \dots, B_n)$ is already known. The proof is finished.

According to Theorem 4, we can remark here that the operation S^n satisfies that superassociative law over the set $P(W_{\tau_n}^{OD_n}(X_n))$.

Let A and B be two subsets of $W_{\tau_n}^{F}(X_n)$. Then we define the binary operation $\hat{+}$ on $P(W_{\tau_n}^{OD_n}(X_n))$ by

$$A \hat{+} B = \hat{S}^n(A, \underbrace{B, \dots, B}_{n}).$$

The following theorem shows that $\hat{+}$ is associative.

Theorem 5 $(P\left(W_{\tau_n}^{OD_n}(X_n)\right), \hat{+})$ is a semigroup.

Proof We show that the binary operation $\hat{+}$ is associative. For this, let A, B, C be arbitrary elements in $P(W_{\tau_n}^{OD_n}(X_n))$. By the superassociativity of the operation S^n, we have

$$(A \hat{+} B) \hat{+} C = \hat{S}^n(A, B, \dots, B) \hat{+} C = \hat{S}^n(\hat{S}^n(A, B, \dots, B), B, \dots, B)$$
$$\hat{S}^n\left(A, \hat{S}^n(B, C, \dots, C), \dots, \hat{S}^n(B, C, \dots, C)\right) = A \hat{+} \hat{S}^n(B, C, \dots, C) = A \hat{+} (B \hat{+} C).$$

Thus, $(P\left(W_{\tau_n}^{OD_n}(X_n)\right), \hat{+})$ is a semigroup.

The obtained semigroup in Theorem 5 is called the *diagonal power semigroup* derived from the power Menger algebra $(P\left(W_{\tau_n}^{OD_n}(X_n)\right), \hat{S}^n)$.

Let A and B be subsets of order-decreasing full terms. For each $i = 1, \dots, n$, the binary operation

$$\cdot_{\{x_i\}} : P(W_{\tau_n}^{OD_n}(X_n)) \times P(W_{\tau_n}^{OD_n}(X_n)) \to P(W_{\tau_n}^{OD_n}(X_n))$$

can be defined by

$$A \cdot_{\{x_i\}} B = \hat{S}^n(A, \{x_1\}, \dots, \{x_{i-1}\}, B, \{x_{i+1}\}, \dots, \{x_n\}).$$

The fact that the binary operation $\cdot_{\{x_i\}}$ satisfies the associative law is now proved.

Theorem 6 $\left(P(W_{\tau_n}^{OD_n}(X_n)), \cdot_{\{x_i\}}\right)$ is a semigroup.

Proof To show that the binary operation $\cdot_{\{x_i\}}$ is associative, let A, B, C be arbitrary subsets of $P(W_{\tau_n}^{OD_n}(X_n))$. Due to the satisfaction of $\hat{S}^n$ of the superassociative law over the set $P(W_{\tau_n}^{OD_n}(X_n))$, we obtain

$$\begin{aligned}(A\cdot_{\{x_i\}}B)\cdot_{\{x_i\}}C &= \hat{S}^n(A,\{x_1\},\dots,\{x_{i-1}\},B,\{x_{i+1}\},\dots,\{x_n\})\cdot_{\{x_i\}}C\\ &= \hat{S}^n(\hat{S}^n(A,\{x_1\},\dots,\{x_{i-1}\},B,\{x_{i+1}\},\dots,\{x_n\}),\{x_1\},\dots,\{x_{i-1}\},C,\{x_{i+1}\},\dots,\{x_n\})\\ &= \hat{S}^n(A,\{x_1\},\dots,\{x_{i-1}\},\hat{S}^n(B,\{x_1\},\dots,\{x_{i-1}\},C,\{x_{i+1}\},\dots,\{x_n\}),\{x_{i+1}\},\dots,\{x_n\})\\ &= A\cdot_{\{x_i\}}\hat{S}^n(B,\{x_1\},\dots,\{x_{i-1}\},C,\{x_{i+1}\},\dots,\{x_n\})\\ &= A\cdot_{\{x_i\}}(B\cdot_{\{x_i\}}C),\end{aligned}$$

which means that $\left(P(W_{\tau_n}{}^{OD_n}(X_n)),\cdot_{\{x_i\}}\right)$ forms a semigroup.

On the Cartesian product $P(W_{\tau_n}{}^{OD_n}(X_n))^n$ of tree languages of order-decreasing full terms of type τ_n, the binary operation

$$\hat{*}: P(W_{\tau_n}{}^{OD_n}(X_n))^n \times P(W_{\tau_n}{}^{OD_n}(X_n))^n \to P(W_{\tau_n}{}^{OD_n}(X_n))^n$$

is defined by

$$(A_1,\dots,A_n)\,\hat{*}\,(B_1,\dots,B_n) = (\hat{S}^n(A_1,B_1,\dots,B_n),\dots,\hat{S}^n(A_n,B_1,\dots,B_n))$$

for all $(A_1,\dots.,A_n),(B_1,\dots.,B_n)\in P(W_{\tau_n}{}^{OD_n}(X_n))^n$.

As a result, we prove:

Theorem 7 $\left(P(W_{\tau_n}{}^{OD_n}(X_n))^n,\hat{*}\right)$ is a semigroup.

Proof Let $(A_1,\dots.,A_n),(B_1,\dots.,B_n),(C_1,\dots.,C_n)\in P(W_{\tau_n}{}^{OD_n}(X_n))^n$. Because a non-deterministic superposition operation satisfies the superassociative law, the equation

$$((A_1,\dots.,A_n)\hat{*}(B_1,\dots.,B_n))\hat{*}(C_1,\dots.,C_n) = (A_1,\dots.,A_n)\hat{*}((B_1,\dots.,B_n)\hat{*}(C_1,\dots.,C_n))$$

is obtained. In fact, we have

$$\begin{aligned}&((A_1,\dots.,A_n)\hat{*}(B_1,\dots.,B_n))\hat{*}(C_1,\dots.,C_n)\\ &= (\hat{S}^n(A_1,B_1,\dots,B_n),\dots,\hat{S}^n(A_n,B_1,\dots,B_n))\,\hat{*}\,(C_1,\dots,C_n)\\ &= (\hat{S}^n(\hat{S}^n(A_1,B_1,\dots,B_n),C_1,\dots,C_n),\dots,\hat{S}^n(\hat{S}^n(A_n,B_1,\dots,B_n),C_1,\dots,C_n))\\ &= (\hat{S}^n\left(A_1,\hat{S}^n(B_1,C_1,\dots,C_n),\dots,\hat{S}^n(B_n,C_1,\dots,C_n)\right),\\ &\qquad \dots,\hat{S}^n\left(A_n,\hat{S}^n(B_1,C_1,\dots,C_n),\dots,\hat{S}^n(B_n,C_1,\dots,C_n)\right))\\ &= (A_1,\dots,A_n)\,\hat{*}\,(\hat{S}^n(B_1,C_1,\dots,C_n),\dots,\hat{S}^n(B_n,C_1,\dots,C_n))\\ &= (A_1,\dots.,A_n)\hat{*}((B_1,\dots,B_n)\hat{*}(C_1,\dots,C_n)).\end{aligned}$$

This shows that $\left(P(W_{\tau_n}{}^{OD_n}(X_n))^n,\hat{*}\right)$ is a semigroup.

We remark here that in this section different kinds of semigroups of terms and tree languages induced by order-decreasing transformations are constructed.

Results

This section is contributed to the embeddability of semigroups derived from the algebra of order-decreasing full terms into semigroups of tree languages of order-decreasing full terms. Recall that we call a semigroup $(S,\cdot)$ can be embedded into a semigroup $(M,\odot)$ if there is a mapping from $(S,\cdot)$ to $(M,\odot)$ which has a compatible property, always knowns as a homomorphism, and such mapping is injective.

Theorem 8 The diagonal semigroup $\left(W_{\tau_n}{}^{OD_n}(X_n),+\right)$ can be embedded into the binary comitant $\left(W_{\tau_n}{}^{OD_n}(X_n)^n,*\right)$.

Proof For any order-decreasing full term t, the mapping $\rho: W_{\tau_n}{}^{OD_n}(X_n)\to W_{\tau_n}{}^{OD_n}(X_n)^n$ can be defined by $\rho(t)=(\underbrace{t,\ldots,t}_{n})$. It is obvious that ρ is injective. To show that, ρ is a homomorphism, we let $s,t\in W_{\tau_n}{}^{OD_n}(X_n)$. Then we have

$$\rho(s+t)=\rho\big(S^n(s,t,\ldots,t)\big)=\Big(\underbrace{S^n(s,t,\ldots,t),\ldots,S^n(s,t,\ldots,t)}_{n}\Big)=(s,\ldots,s)+(t,\ldots,t)=\rho(s)+\rho(t).$$

Therefore, $\left(W_{\tau_n}{}^{OD_n}(X_n),+\right)$ can be embedded into $\left(W_{\tau_n}{}^{OD_n}(X_n)^n,*\right)$.

Theorem 9 The power diagonal semigroup $(P\left(W_{\tau_n}^{OD_n}(X_n)\right),\hat{+})$ can be embedded into $\left(P(W_{\tau_n}{}^{OD_n}(X_n))^n,\hat{*}\right)$.

Proof We define the mapping $\gamma: P(W_{\tau_n}{}^{OD_n}(X_n))\to P(W_{\tau_n}{}^{OD_n}(X_n))^n$ by $\gamma(A)=(\underbrace{A,\ldots,A}_{n})$ for all $A\in P(W_{\tau_n}{}^{OD_n}(X_n))$. Clearly, γ is an injective mapping. Furthermore, we have

$$\gamma(A\hat{+}B)=\gamma\left(\hat{S}^n(A,B,\ldots,B)\right)=\left(\underbrace{\hat{S}^n(A,B,\ldots,B),\ldots,\hat{S}^n(A,B,\ldots,B)}_{n}\right)=(A,\ldots,A)\hat{+}(B,\ldots,B)=\gamma(A)\hat{+}\gamma(B).$$

We conclude that γ is a monomorphism from $(P\left(W_{\tau_n}^{OD_n}(X_n)\right),\hat{+})$ to $\left(P(W_{\tau_n}{}^{OD_n}(X_n))^n,\hat{*}\right)$.

To close this work, we prove that the semigroups of order-decreasing full terms can be embedded into the semigroups of tree languages of order-decreasing full terms.

Theorem 10 The following statements are ture:

(1) $(W_{\tau_n}{}^{OD_n}(X_n),S^n)$ is embeddable into $(P\left(W_{\tau_n}^{OD_n}(X_n)\right),\hat{S}^n)$,

(2) $\left(W_{\tau_n}{}^{OD_n}(X_n),+\right)$ is embeddable into $(P\left(W_{\tau_n}^{OD_n}(X_n)\right),\hat{+})$,

(3) $\left(W_{\tau_n}{}^{OD_n}(X_n),\cdot_{x_i}\right)$ is embeddable into $\left(P(W_{\tau_n}{}^{OD_n}(X_n)),\cdot_{\{x_i\}}\right)$,

(4) $\left(W_{\tau_n}{}^{OD_n}(X_n)^n,*\right)$ is embeddable into $\left(P(W_{\tau_n}{}^{OD_n}(X_n))^n,\hat{*}\right)$.

Proof We first show that the statement (1) holds. To do this, we define the mapping η from $(W_{\tau_n}^{OD_n}(X_n), S^n)$ to $(P\left(W_{\tau_n}^{OD_n}(X_n)\right), \hat{S}^n)$ by $\eta(t) = \{t\}$ for all order-decreasing full term t of type τ_n. Clearly, η is injective. Moreover, a homomorphism property, i.e., $\eta\left(S^n(t, s_1, \dots, s_n)\right) = \hat{S}^n(\eta(t), \eta(s_1), \dots, \eta(s_n))$ is also valid. In fact, we give a proof by induction on the complexity of an order-decreasing full term t. If $t = f_i(x_{\alpha(1)}, \dots, x_{\alpha(n)})$ where $\alpha \in OD_n$, we have $\eta(S^n(f_i(x_{\alpha(1)}, \dots, x_{\alpha(n)}), s_1, \dots, s_n)) = \eta(f_i(s_{\alpha(1)}, \dots, s_{\alpha(n)})) = \{f_i(s_{\alpha(1)}, \dots, s_{\alpha(n)})\}$ and

$$\hat{S}^n\left(\eta\left(f_i\left(x_{\alpha(1)}, \dots, x_{\alpha(n)}\right)\right), \eta(s_1), \dots, \eta(s_n)\right) = \hat{S}^n(\{f_i(x_{\alpha(1)}, \dots, x_{\alpha(n)})\}, \{s_1\}, \dots, \{s_n\}) = \{f_i(s_{\alpha(1)}, \dots, s_{\alpha(n)})\}.$$

This shows that the equation

$$\eta\left(S^n\left(f_i\left(x_{\alpha(1)}, \dots, x_{\alpha(n)}\right), s_1, \dots, s_n\right)\right) = \hat{S}^n(\eta\left(f_i\left(x_{\alpha(1)}, \dots, x_{\alpha(n)}\right)\right), \eta(s_1), \dots, \eta(s_n))$$

holds. If $t = f_i(t_1, \dots, t_n)$ and assume that $\eta\left(S^n\left(t_j, s_1, \dots, s_n\right)\right) = \hat{S}^n(\eta(t_j), \eta(s_1), \dots, \eta(s_n))$ for every $1 \le j \le n$, we have

$$\eta(S^n(f_i(t_1, \dots, t_n), s_1, \dots, s_n)) = \{S^n(f_i(t_1, \dots, t_n), s_1, \dots, s_n)\} = \{f_i(S^n(t_1, s_1, \dots, s_n), \dots, S^n(t_n, s_1, \dots, s_n))\} \text{ and}$$

$$\begin{aligned}\hat{S}^n\left(\eta(f_i(t_1, \dots, t_n)), \eta(s_1), \dots, \eta(s_n)\right) &= \hat{S}^n(\{f_i(t_1, \dots, t_n)\}, \{s_1\}, \dots, \{s_n\}) \\ &= \left\{f_i(r_1, \dots, r_n) | r_j \in \hat{S}^n(\{t_j\}, \{s_1\}, \dots, \{s_n\}), j = 1, \dots, n\right\}.\end{aligned}$$

Thus,

$\eta\left(S^n\left(f_i(t_1, \dots, t_n), s_1, \dots, s_n\right)\right) = \hat{S}^n(\eta\left(f_i(t_1, \dots, t_n)\right), \eta(s_1), \dots, \eta(s_n))$. Consequently, η is a monomorphism from the Menger algebra $(W_{\tau_n}^{OD_n}(X_n), S^n)$ to the algebra $(P\left(W_{\tau_n}^{OD_n}(X_n)\right), \hat{S}^n)$.

To prove the statements (2) and (3), we define the mapping

$$\beta : W_{\tau_n}^{OD_n}(X_n) \to P(W_{\tau_n}^{OD_n}(X_n))$$

by

$$\beta(t) = \{t\}$$

for all $t \in W_{\tau_n}^{OD_n}(X_n)$. Obviously, β is an injection. It is not difficult to verify that the following equations are satisfied: $\beta(s + t) = \beta(s) + \beta(t)$ and $\beta(s \cdot_{x_i} t) = \beta(s) \cdot_{\{x_i\}} \beta(t)$.

Finally, we prove that (4) is valid. Let $(t_1, \dots, t_n)$ be a product of $W_{\tau_n}^{OD_n}(X_n)^n$. The mapping $\varphi : W_{\tau_n}^{OD_n}(X_n)^n \to P(W_{\tau_n}^{OD_n}(X_n)^n)^n$ can be defined by

$$\varphi((t_1, \dots, t_n)) = (\{t_1\}, \dots, \{t_n\}).$$

It is clear that φ is injective. The proof of the equation

$$\varphi((t_1, \dots, t_n) * (s_1, \dots, s_n)) = \varphi((t_1, \dots, t_n)) \hat{*} \varphi((s_1, \dots, s_n))$$

is omitted.

Discussion

We remark that all semigroups derived from the algebra of order-decreasing full terms obtained in Theorems 1-3 can be viewed as a specific case of semigroups induced by the Menger algebra of tree languages of order-decreasing full terms given in Theorems 5-7 because a non-deterministic operation $\hat{S}^n$ maps from the set of tree languages of order-decreasing full terms into itself, while an operation S^n sends from the set of order-decreasing full terms into itself. For this reason, it is possible to describe monomorphisms of these structures.

Conclusions

In this work, based on the concept of order-decreasing full terms, different binary operations are defined. We also prove that these operations are associative and hence the corresponding semigroups are obtained. In general, we consider a power set of $W_{\tau_n}{}^{OD_n}(X_n)$ of order-decreasing full terms whose elements are called tree languages of order-decreasing full terms. The fact that a non-deterministic superposition $\hat{S}^n$ on $P(W_{\tau_n}{}^{OD_n}(X_n))$ satisfies the superassociativity is proved in Theorem 5. This leads us to construct semigroups of tree languages of order-decreasing full terms under the operation $+, \cdot_{\{x_i\}}$ and $\hat{*}$. Finally, embedding theorems of the semigroups of order-decreasing full terms into semigroups of tree languages constructed from order-decreasing full terms are discussed.

Acknowledgment

This work was supported by Rajamangala University of Technology Rattanakosin, Thailand.

References

Chansuriya, N. (2021). All maximal idempotent submonoids of generalized cohypersubstitutions of type $\tau = (2)$, *Discuss. Math., Gen. Algebra Appl.*, *41*(1), 45-54.

Denecke, K., Hounnon, H. (2021). Partial Menger algebras of terms, *Asian-Eur. J. Math.*, *14*(6) 2150092.

Denecke, K. (2016). The partial clone of linear terms, *Siberian Math. J. 57*(4), 589-598.

Dudek, W.A. , Trokhimenko, V.S. (2018). Menger algebras of associative and self-distributive *n*-ary operations. *Quasigroups Relat. Syst.*, *26*, 45-52.

Dudek, W.A. , Trokhimenko, V.S. (2021). Menger algebras of *k*-commutative *n*-place functions, *Georgian Math. J.*, *28*(3), 355-361.

Joomwong, J., Phusanga, D. (2021). On Green's relations which are related to an algebraic system of type $((n),(m))$, *Southeast Asian Bull. Math.*, *45*(6), 897-904.

Kumduang, T., Leeratanavalee, S. (2021). Menger hyperalgebras and their representations. *Commun. Algebra.*, *49*, 1513-1533.

Kumduang, T.; Leeratanavalee, S. (2021). Semigroups of terms, tree languages, Menger algebra of *n*-ary functions and their embedding theorems, *Symmetry*, *13*(4), 558.

Kumduang, T., Sriwongsa, S. (2022). A note on superassociative algebra of terms determined by singular mappings on a fnite set, *Int. J. Math. Comput. Sci.*, *14*(4), 1541-1546.

Phuapong, S., Pookpienlert, C. (2022). $\overline{S}(n_i,Y_i)$)-terms and their algebraic properties, *Thai J. math.*, *20*(1), 337-346.

Sun, L. (2020). A natural partial order on partition order-decreasing transformation semigroups, *Bulletin of the Iranian Mathematical Society*, *46*, 1357-1369.

Umar, A. (1992). On the semigroups of order-decreasing finite full transformations. *Proc. R. Soc. Edinb. A, 120,* 129-142.

Wattanatripop, K., Changphas, T. (2020). Clones of terms of a fxed variable, *Mathematics*, *8,* 260.

Wattanatripop, K., Changphas, T. (2021). On $nd-K^{*}(n,r)$-Full hypersubstitutions, *Discuss. Math., Gen. Algebra Appl.*, *41*(2), 213–227.

Wattanatripop, K., Changphas, T. (2019). The clone of $K^*(n,r)$-full terms, *Discuss. Math., Gen. Algebra Appl.*, *39*(2), 277-288.

Wattanatripop, K., Changphas, T. (2021). The Menger algebra of terms induced by order-decreasing transformations, *Commun. Algebra*, *49*(7), 3114-3123.

Yang, H.B., Yang, X.L. (2012). Automorphisms of partition order-decreasing transformation monoids. *Semigroup Forum, 85*(3), 513–524.