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การหาผลเฉลยของสมการไดโอแฟนไทน์ $x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2 = u^2$ โดยวิธีทางเรขาคณิต

A Geometrial Approach to the Diophantine Equation $x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2 = u^2$

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บทคัดย่อ

เราหาผลเฉลยทั้งหมดของสมการไดโอแฟนไทน์ $x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2 = u^2$ โดยพัฒนาจากวิธีทางเรขาคณิตของ (Ayoub, 1984) ที่ใช้ในการหาผลเฉลยของสมการ $x^2 + y^2 + z^2 = u^2$ เราสามารถหาจุดตรรกยะทั้งหมดบนทรงกลมหนึ่งหน่วย n มิติ ผ่านการลากเส้นตรงระหว่างจุดตรรกยะเหล่านั้นกับจุด $(1, 0, \ldots, 0)$ ซึ่งสมการอิงตัวแปรเสริมเส้นตรงดังกล่าวจะมีความชันเป็นตรรกยะเสมอ

คำสำคัญ : สมการไดโอแฟนไทน์ ; สามสิ่งอันดับพีทาโกรัส ; ทรงกลมหนึ่งหน่วย n มิติ

Abstract

We find all Diophantine solutions for the equation $x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2 = u^2$ by refining the geometrical approach from (Ayoub, 1984) to find solutions of the equation $x^2 + y^2 + z^2 = u^2$. We can find all rational points on the unit n-sphere by lines connecting those rational points to the point $(1, 0, \ldots, 0)$. Such linear parametric equations will always have rational slopes.

Keywords : Diophantine solutions ; Pythagorean triples ; unit n – sphere

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Introduction

The simplest form of the equation $x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2 = u^2$ is Pythagorean equation $x_1^2 + x_2^2 = u^2$, whose all Diophantine solutions are the Pythagorean triples $(m^2 - n^2,\ 2mn,\ m^2 + n^2)$, where m, n are integers. It is a far more difficult problem in the case of more variables. For example, the equation $x_1^2 + x_2^2 + x_3^2 = u^2$ is studied by many others. The author of (Dickson, 1966) has mentioned many classes of general Diophantine solutions given by (Euler, 1779), (Cossali, 1797), (Gill, 1826), (Lebesgue, 1874), (Dainelli, 1877), (Catalan, 1885), and (Mikami, 1912), respectively as follows for any integers m, n, p, q, r :

1. $(4mp)^2 + [(m^2-1)(p^2+1)]^2 + [2m(p^2-1)]^2 = [(m^2+1)(p^2+1)],^2$ in (Euler, 1779).
2. $q^2 + (q+1)^2 + [q(q+1)]^2 = (q^2+q+1)^2$, in (Cossali, 1797).
3. $[2qr(m^2-n^2)]^2 + [(m^2-n^2)(q^2-r^2)]^2 + [2mn(q^2+r^2)]^2 = [(m^2+n^2)(q^2+r^2)]^2$, in (Gill, 1826).
4. $(2pr)^2 + (2qr)^2 + (p^2+q^2-r^2)^2 = (p^2+q^2+r^2)^2$, in (Lebesgue, 1874).
5. $[p(p+q)]^2 + [q(p+q)]^2 + (pq)^2 = (p^2+pq+p^2)^2$, in (Dainelli, 1877).
6. $(p^2+q^2-r^2-s^2)^2 + [2(pr+qs)]^2 + [2(ps-qr)]^2 = (p^2+q^2+r^2+s^2)^2$, in (Catalan, 1885).
7. $(4m^2n^2)^2 + (m^4-n^4)^2 + [2mn(m^2-n^2)]^2 = [(m^2+n^2)^2]^2$, in (Mikami, 1912).

However, among the above forms, only Form 6 produces complete solutions.

In (Ayoub, 1984), Ayoub has studied a geometrical approach to the problem by observing the obvious solutions from the Pythagorean triples $(m^2 - n^2,\ 2mn,\ 0,\ m^2 + n^2)$. Then he has constructed parametric equations with integral slopes containing known solutions. Such equations intersect a 3-sphere centered at the origin of radius u and provide a class of solutions including all previously mentioned six classes. However, the solutions are still incomplete. As the ending note, Ayoub has mentioned that one can use the same technique for an even more general case. His work relates to more studies on primitive cuboids, for example, in (Huerlimann, 2002) and (Huerlimann, 2015).

In this article, we intend to find all Diophantine solutions to the equation $x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2 = u^2$. Solving the equation directly is very difficult even in 3 variables, as in previously mentioned researches, so we tackle the problem similarly to (Ayoub, 1984). Instead of using a parametric equation with integral slopes from many known solutions, we consider linear parametric equations with rational slopes connecting to just one obvious solution, $(u, 0, \ldots, 0)$. This refinement actually gives all Diophantine solutions to the generalized problem.

Methods

Diophantine solutions $(x_1, x_2, \ldots, x_n)$ to the equation $x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2 = u^2$ are exactly rational solutions $\left(\frac{x_1}{u}, \frac{x_2}{u}, \ldots, \frac{x_n}{u}\right)$ to the equation

$$x_1'^2 + x_2'^2 + \ldots + x_n'^2 = 1. \quad (1)$$

Consequently, it is equivalent to locate all rational points on the unit n – sphere. We shall solve the question by radiating lines with rational slopes from a trivial solution $Q = (1, 0, \ldots, 0)$ on the sphere and illustrate that they intersect the sphere at all other rational solutions.

Results

First we will clarify the method. A point $P = \left(\frac{r_1}{s_1}, \frac{r_2}{s_2}, \ldots, \frac{r_n}{s_n}\right) \in \mathbb{R}^n$ is called a rational point if $\frac{r_i}{s_i}$'s are rational numbers. For each rational point P, the line L containing P and $Q = (1, 0, \ldots, 0)$ is the parametric equation $Q(1-t) + Pt$. To be precise, the equation for L is given by

$$x_1 = 1 + \left(\frac{r_1}{s_1} - 1\right)t, \text{ and } x_i = \frac{r_i}{s_i}t, \quad (2)$$

for $i = 2, 3, \ldots, n$. Therefore, P and Q can be located by letting $t = 1$, and $t = 0$ respectively. For example, if $n = 2$, we illustrate locations of P and Q on L as in Figure 1.

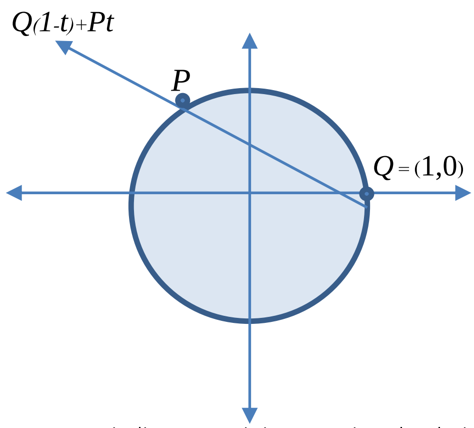

Figure 1 The parametric line containing a rational solution in two-dimension.

We need to show that the lines containing Q with rational slopes intersects the unit n – sphere at rational points, and likewise, any rational point P on the sphere is connected to Q by a line with rational slopes.

Lemma 1. Let P be a point on the unit $n-$ sphere. Then P is a rational point if and only if the linear parametric equation containing P and Q has the form

$$x_1' = 1 + \frac{r_1}{s_1}t,\ x_2' = \frac{r_2}{s_2}t,\ \dots,\ x_n' = \frac{r_n}{s_n}t \tag{3}$$

for some rational numbers $\frac{r_i}{s_i}$'s.

Proof. Suppose that P is a rational point. Then $P = \left(\frac{r_1}{s_1}, \frac{r_2}{s_2}, \dots, \frac{r_n}{s_n}\right)$. We recall that the line connecting P and Q is given by Equation (2), whose slopes are already rational numbers.

Now let P be contained in a line

$$x_1' = 1 + \frac{r_1}{s_1}t,\ x_2' = \frac{r_2}{s_2}t,\ \dots,\ x_n' = \frac{r_n}{s_n}t \tag{4}$$

for some rational numbers $\frac{r_i}{s_i}$'s. Notice that at $t = 0$ lies the point Q, so we can safely assume that $t \neq 0$ to find P, which is the other intersection point with the $n-$ sphere. We solve

$$\begin{aligned} 1 &= \left(1 + \frac{r_1}{s_1}t\right)^2 + \left(\frac{r_2}{s_2}t\right)^2 + \dots + \left(\frac{r_n}{s_n}t\right)^2 \\ -2\frac{r_1}{s_1}t &= \left(\frac{r_1}{s_1}t\right)^2 + \left(\frac{r_2}{s_2}t\right)^2 + \dots + \left(\frac{r_n}{s_n}t\right)^2 \\ t &= \frac{-2\frac{r_1}{s_1}}{\left(\frac{r_1}{s_1}\right)^2 + \left(\frac{r_2}{s_2}\right)^2 + \dots + \left(\frac{r_n}{s_n}\right)^2}. \end{aligned} \tag{5}$$

where v is the rational number

$$\left(\frac{r_1}{s_1}\right)^2 + \left(\frac{r_2}{s_2}\right)^2 + \dots + \left(\frac{r_n}{s_n}\right)^2 = \sum_{i=1}^{n}\left(\frac{r_i}{s_i}\right)^2, \tag{6}$$

and substitute t in Equation (4) to establish that

$$x_1' = 1 + \frac{r_1}{s_1} \cdot \frac{-2r_1}{s_1 v} = 1 - \frac{2r_1^2}{s_1^2 v}, \quad \text{and}$$

$$x_i' = \frac{r_i}{s_i} \cdot \frac{-2r_1}{s_1 v} = 1 - \frac{2r_1 r_i}{s_1 s_i v}, \tag{7}$$

for $i = 2, 3, \dots, n$. This proves that P must be a rational point.

With Lemma 1, we are ready to prove the main result.

Theorem 1. All Diophantine solutions to the equation $x_1^2 + x_2^2 + x_3^2 + \dots + x_n^2 = u^2$ are

$$\left[\sum_{i=1}^{n}\left(\frac{r_i s}{s_i}\right)^2\right]^2 = \left[-\left(\frac{r_1 s}{s_1}\right)^2 + \sum_{i=2}^{n}\left(\frac{r_i s}{s_i}\right)^2\right]^2 + \left(\frac{2r_1 s}{s_1}\right)^2 \cdot \sum_{i=2}^{n}\left(\frac{r_i s}{s_i}\right)^2, \tag{8}$$

where $\frac{r_i}{s_i}$'s are rational numbers and $s = s_1 s_2 \dots s_n$.

Proof. We have already known that any Diophantine solution of this equation can be obtained from finding a rational solution of the equation $x_1'^2 + x_2'^2 + x_3'^2 + \dots + x_n'^2 = 1$. By Lemma 1, each line passing through $Q = (1, 0, 0, \dots, 0)$ with rational slopes $\frac{r_i}{s_i}$'s result to a rational solution

$$\left(1 - \frac{2r_1^2}{s_1^2 v}\right)^2 + \left(\frac{2r_1 r_2}{s_1 s_2 v}\right)^2 + \dots + \left(\frac{2r_1 r_n}{s_1 s_n v}\right)^2 = 1. \tag{9}$$

Recalling that $v = \sum_{i=1}^{n}\left(\frac{r_i}{s_i}\right)^2$, one can rearrange the above equation to

$$\begin{aligned} v^2 &= \left(v - \frac{2r_1^2}{s_1^2}\right)^2 + \left(\frac{2r_1 r_2}{s_1 s_2}\right)^2 + \dots + \left(\frac{2r_1 r_n}{s_1 s_n}\right)^2 \\ &= \left[-\left(\frac{r_1}{s_1}\right)^2 + \left(\frac{r_2}{s_2}\right)^2 + \dots + \left(\frac{r_n}{s_n}\right)^2\right]^2 + \left(\frac{2r_1 r_2}{s_1 s_2}\right)^2 + \dots + \left(\frac{2r_1 r_n}{s_1 s_n}\right)^2 \qquad (10) \\ &= \left[-\left(\frac{r_1}{s_1}\right)^2 + \sum_{i=1}^{n}\left(\frac{r_i}{s_i}\right)^2\right]^2 + \left(\frac{2r_1}{s_1}\right)^2 \cdot \sum_{i=2}^{n}\left(\frac{r_i}{s_i}\right)^2. \end{aligned}$$

To clear out denominators, let $s = s_1 s_2 \ldots s_n$ and multiply the above equation by s^4 to get

$$\left[\sum_{i=1}^{n}\left(\frac{r_i s}{s_i}\right)^2\right]^2 = \left[-\left(\frac{r_1 s}{s_1}\right)^2 + \sum_{i=2}^{n}\left(\frac{r_i s}{s_i}\right)^2\right]^2 + \left(\frac{2 r_1 s}{s_1}\right)^2 \cdot \sum_{i=2}^{n}\left(\frac{r_i s}{s_i}\right)^2. \qquad (11)$$

Since $s_i \mid s$ for each $i = 1, 2, 3, \ldots, n,$ we have finally found all Diophantine solutions as required.

Discussion

The key to our result is that a line connecting two rational points in the unit $n -$ sphere always has rational slopes. By knowing one obvious rational point $Q = (1, 0, \ldots, 0)$ on the sphere, one can safely find all rational points, which leads to all Diophantine solutions to the equation $x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2 = u^2$. To prove slightly further, given each set of rational slopes $\left(\frac{r_1}{s_1}, \frac{r_2}{s_2}, \ldots, \frac{r_n}{s_n}\right)$, we can construct a line passing through Q with these slopes. When it is not a tangent line, it intersects the unit $n -$ sphere at the other rational point. Could we, perhaps determine whether two sets of rational slopes result to a duplicate solution?

The answer is yes, and is quite simple. A line equation is determined by two points, and one is already Q. Hence, different solutions must result from different lines. This implies that a duplicate solution only occurs when two sets of slopes are equal. By requiring the slopes to be irreducible fractions, we can conclude that each $\left(\frac{r_1}{s_1}, \frac{r_2}{s_2}, \ldots, \frac{r_n}{s_n}\right)$ produces a unique Diophantine solution.

Conclusions

We finish the article by emphasizing that this geometrical approach to solve the Diophantine equation $x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2 = u^2$ is elegant in a way that it requires only fundamental methods to solve a heavy problem. Although our approach is based on the work of (Ayoub, 1984), Ayoub's result must rely on known solutions to construct more solutions in a higher dimension. The difficulty would rise and still does not provide complete solutions, while our modified method, by introducing rational slopes, only relies on one obvious rational solution to solve for the complete list.

For any dimension, the point $Q = (1, 0, \ldots, 0)$ is always a rational point on the unit $n -$ sphere. In fact, one can choose Q to be any rational point on the unit $n -$ sphere to obtain the same result, because the linear parametric equation containing two rational points always has rational slopes. Such a line, that is not a tangent line, will always intersect the sphere at another solution. This provides a complete list of solutions in Equation (11), which includes all results discussed in the first section.

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